Recall: variation of GIT (in toric case)

last time: • makes sense to talk about $\mathbb{Q}$-polarization.

$H \overset{i}{\hookrightarrow} T \rightsquigarrow i^*: M_T \to M_H$

$$i^*_{\mathbb{R}}: \begin{array}{ccc} M_{T,\mathbb{R}} & \supseteq & Q \\ \downarrow & & \downarrow \\ M_{H,\mathbb{R}} & \supseteq & i^*_{\mathbb{R}} Q \end{array}$$

(X, L)

X: T-toric

L: T-linearized

$\Updownarrow$

$M_{T,\mathbb{R}} \supseteq Q$ polytope.

$M_{T,\mathbb{R}} = M_T \otimes_{\mathbb{Z}} \mathbb{R}$

$H \curvearrowright L$! can be changed

$\downarrow$

$H \curvearrowright X$

$\Updownarrow$

variation of the linearization.

$\Downarrow$

different $X/\!/_c H$

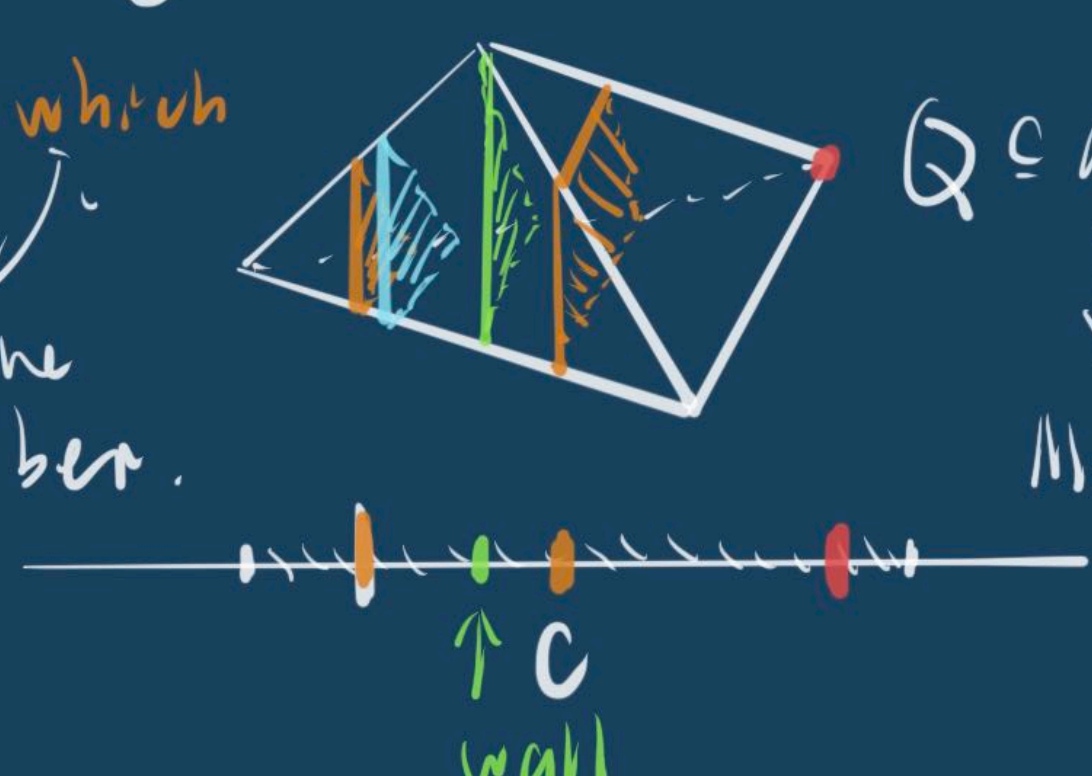
change the linearization

$\Updownarrow$

change of the character.

$X/\!/_c H \longleftrightarrow$ polytope. ⟸ which ?

Answer: the fiber.



$Q \subseteq M_{T,\mathbb{R}}$

$\downarrow i^*_{\mathbb{R}}$

$M_{H,\mathbb{R}}$

• For generic c, in a small nbd the normal fan doesn't change.

i.e. $c' \in$ small nbd of c.

$\Rightarrow X/\!/_c H \cong X/\!/_{c'} H$

• For special c, (this) not true, we have a birational contraction $X/\!/_{c'} H \to X/\!/_c H$, for any c' in a small nbd of c.

• For boundary c

$X/\!/_{c'} H \to X/\!/_c H$ has +dim fibers.

• $c \notin i^*_{\mathbb{R}} Q \Rightarrow X/\!/_c H = \phi$.

The phenomenon is called wall crossing.

Thm (Dolgachev-Hu, Thaddeus)

$\exists$ only finitely many walls (chambers) for VGIT.

(not only for toric case).

e.g. $\mathbb{P}^1_x \times \mathbb{P}^1_y \times \mathbb{P}^1_z \,/\, \mathbb{C}^*$ $\quad L = \mathcal{O}(1,1,1)$

$$\left.\begin{aligned} \mathbb{C}^* \ni \lambda \cdot [x_0 : x_1] &= [x_0 : \lambda x_1] \\ \lambda \cdot [y_0 : y_1] &= [y_0 : \lambda y_1] \\ \lambda \cdot [z_0 : z_1] &= [z_0 : \lambda z_1] \end{aligned}\right\} (*)$$

What is the VGIT quotient?

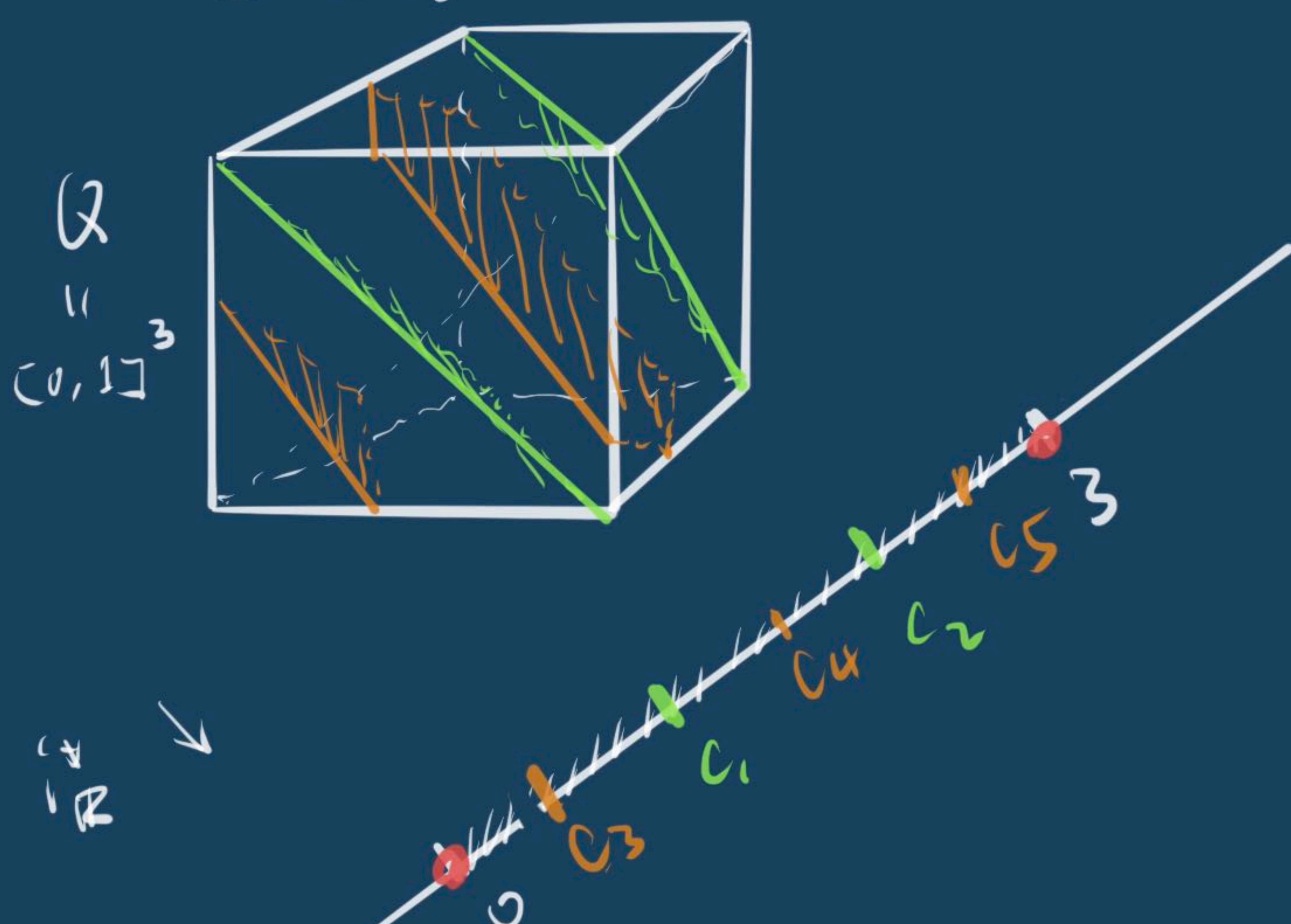
$X = (\mathbb{P}_1)^3$

$\mathbb{C}^* = H \overset{i}{\hookrightarrow} T = (\mathbb{C}^*)^3$

by $(*)$ $\lambda \mapsto (\lambda, \lambda, \lambda)$

$i^* : M_T \to M_H \quad (\otimes \mathbb{R})$

$(a, b, c) \to a + b + c$



$X /\!/_{c_1} H = \mathbb{P}^2 \qquad X /\!/_{c_2} H = \mathbb{P}^2$

$\updownarrow \qquad\qquad \updownarrow$

$X /\!/_{c_3} H \cong X /\!/_{c_5} H \cong \mathbb{P}^2$

$X /\!/_{c_4} H \cong Bl_3 \mathbb{P}^2$

$\updownarrow$

$X /\!/_0 H \cong X /\!/_3 H = pt.$

Quotient construction of toric var's.

e.g. $\mathbb{P}^n = (\mathbb{A}^{n+1} - \{0\}) \,/\, \mathbb{C}^*$

$= \mathbb{A}^{n+1} /\!/ \mathbb{C}^*$

replaced by any toric var's

[?]

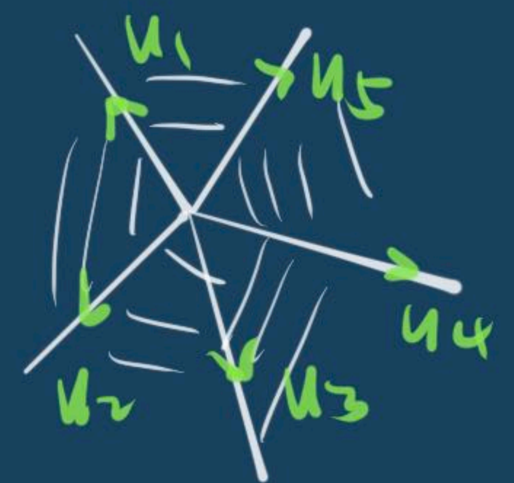
We do have such an analogue.

$X = X_\Sigma$, Σ fan in $N_{\mathbb{R}}$

T-toric ↘ spanned by u_ρ : primitive vectors $\rho \in \Sigma(1)$.

↕

D_ρ : prime toric div's.



Recall: $0 \to M \to \mathbb{Z}^{\Sigma(1)} \to Cl(X) \to 0$ (linear equiv.)

Apply: $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$:

$$1 \to \mathrm{Hom}_{\mathbb{Z}}(Cl(X), \mathbb{C}^*) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \to 1$$

$$\mathrm{Hom}_{\mathbb{Z}}(Cl(X), \mathbb{C}^*) =: G, \quad \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \cong (\mathbb{C}^*)^{\Sigma(1)}, \quad \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \cong T_N$$

i.e. $1 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \to T_N \to 1$. (*)

view $G \subseteq (\mathbb{C}^*)^{\Sigma(1)}$ as a subgp

Note: $Cl(X) = \mathbb{Z}^l \times H$, H finite ab. gp.

Then: $G = \mathrm{Hom}_{\mathbb{Z}}(Cl(X), \mathbb{C}^*)$

$= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^l \times H, \mathbb{C}^*)$

$= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^l, \mathbb{C}^*) \times \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{C}^*)$

$= (\mathbb{C}^*)^l \times \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{C}^*)$

So $G \cong$ torus $\times$ finite ab. gp.

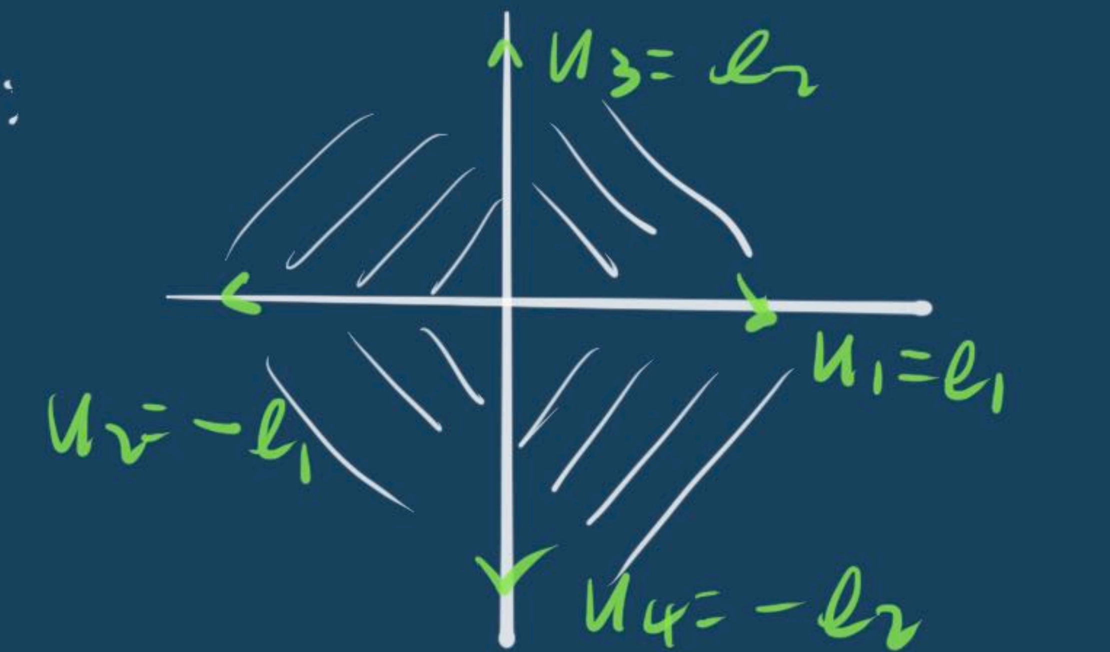
($\Rightarrow G$ is reductive)

Explicitly, by (*)

$$G = \left\{ (t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \;\middle|\; \prod_{\rho \in \Sigma(1)} t_\rho^{\langle m, u_\rho \rangle} = 1 \right\} \quad \forall m \in M.$$

e.g. $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$, $N \cong \mathbb{Z}^2$

fan:



$t = (t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$

$$G \ni t \Leftrightarrow 1 = t_1^{\langle m, e_1 \rangle} t_2^{\langle m, -e_1 \rangle} t_3^{\langle m, e_2 \rangle} \cdot t_4^{\langle m, -e_2 \rangle}, \forall m \in M$$

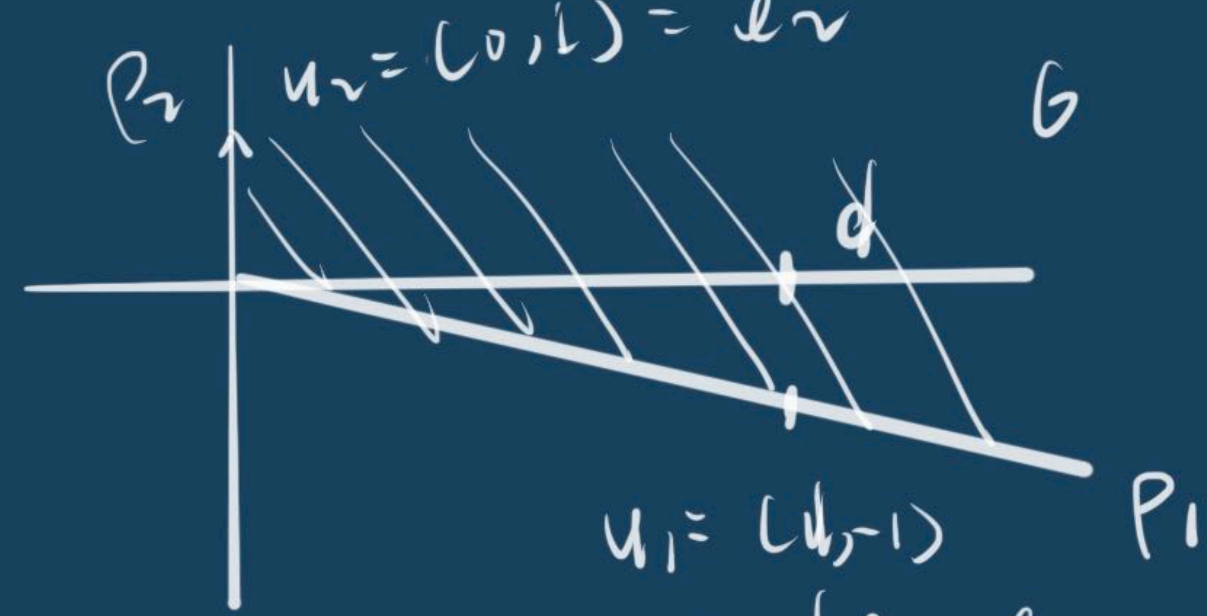
Take $m = e_1^*, e_2^*$, get

$t_1 t_2^{-1} = 1$, $t_3 t_4^{-1} = 1$

$t_1 = t_2$, $t_3 = t_4$

$$\Rightarrow G = \{ (\mu, \mu, \lambda, \lambda) \in (\mathbb{C}^*)^4 \} \cong (\mathbb{C}^*)^2$$

e.g.



$$0 = \operatorname{div}(\chi^{e_1}) = \langle e_1, u_1 \rangle D_1 + \langle e_1, u_2 \rangle D_2 = (1,0)(d,-1)D_1 + (1,0)(0,1)D_2 = dD_1$$

$$0 = \operatorname{div}(\chi^{e_2}) = \langle e_2, u_1 \rangle D_1 + \langle e_2, u_2 \rangle D_2 = (0,1)(d,-1)D_1 + (0,1)(0,1)D_2 = -D_1 + D_2$$

$\rightsquigarrow$ $D_1 = D_2$, $dD_1 = 0$

$\Rightarrow$ $Cl(X_\sigma)$ is gen. by D_1, relation $d[D_1] = 0$.

$\Rightarrow$ $Cl(X_\sigma) \cong \mathbb{Z}/d\mathbb{Z}$

$\Rightarrow$ $G = \operatorname{Hom}_{\mathbb{Z}}(Cl(X_\sigma), \mathbb{C}^*) \cong \mu_d$

Cox idea: to get X_Σ as a quotient

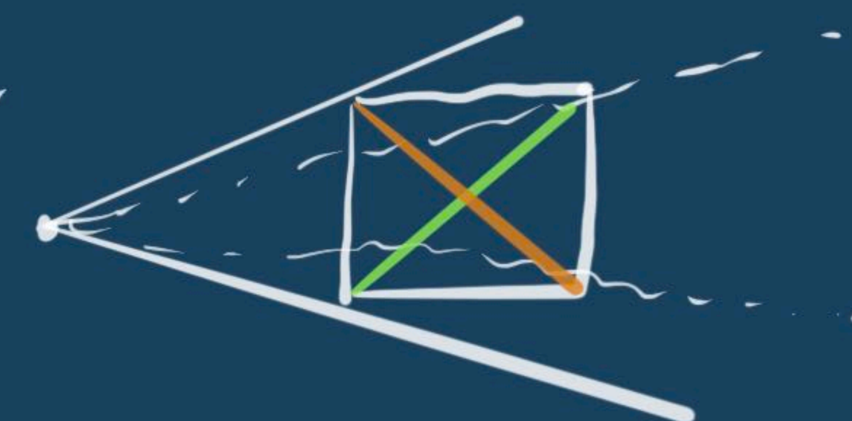
need: · G

· remove "bad pts"

now: G ✓ $\Sigma(1)$ ✓

miss the information about the fan Σ.

e.g.



fix four rays

$\Downarrow$

Cone $(\mathbb{P}^1 \times \mathbb{P}^1)$

Cone $(\mathbb{P}^2 \cup \mathbb{P}^2)$

$(\mathbb{P}^2 \cup \mathbb{P}^2)$

Def: $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ is called the total coordinate ring of X_Σ

for each cone $\sigma \in \Sigma$

$$x^{\hat{\sigma}} := \prod_{\rho \notin \sigma(1)} x_\rho \qquad S \supseteq$$

gen. an ideal: $B(\Sigma) = \langle x^{\hat{\sigma}}, \sigma \in \Sigma \rangle$

which is called the irrelevant ideal (or Stanley-Reisner ideal).

Rmk: actually one may only look at max cones, i.e.

$$B(\Sigma) = \langle x^{\hat{\sigma}}, \sigma \text{ max. in } \Sigma \rangle$$

Benefit: $\Sigma(1)$ together w/ $B(\Sigma)$

$\downarrow$ determines

Σ

"bad" pts : $Z(\Sigma) = V(B(\Sigma))$

$\cap\!\mid$

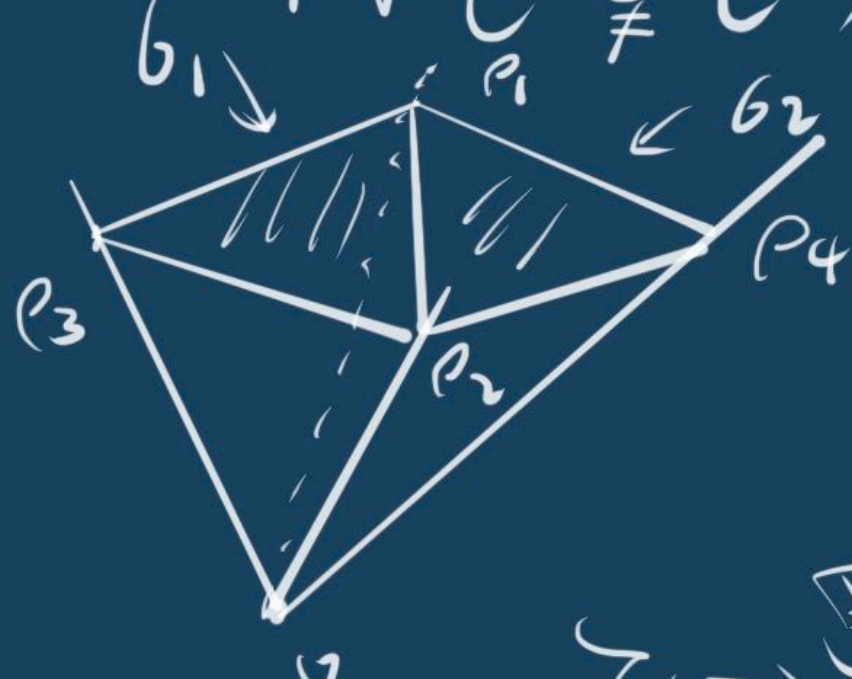
$\mathbb{C}^{\Sigma(1)}$

Q: What is $Z(\Sigma)$?

A: Def: A primitive collection $C \subseteq \Sigma(1)$ satisfies

- $C \not\subseteq \sigma(1)$, $\forall \sigma \in \Sigma$
- $\forall C' \subsetneq C$, $\exists\, \sigma \in \Sigma$ s.t. $C' \subseteq \sigma(1)$.

e.g.



$C = \{\rho_1, \rho_2, \rho_3, \rho_4\}$ (← remove ρ_1)

or $C = \{\rho_2, \rho_3, \rho_4\}$.

The answer: $Z(\Sigma) \overset{(*)}{=} \bigcup_{\substack{C \text{ primitive} \\ \text{in } \Sigma(1)}} V(x_\rho,\ \rho \in C)$.

e.g. $\mathbb{P}^1 \times \mathbb{P}^1$

cone	irr. ideal
σ_1	x_2x_4
σ_2	x_1x_4,
σ_3	x_1x_3,
σ_4	x_2x_3

$\langle x_2x_4, x_1x_4, x_1x_3, x_2x_3 \rangle = B(\Sigma)$

$$Z(\Sigma) = V(B(\Sigma)) = \mathbb{C}^2_{x_1,x_2} \times \{0\} \cup \{0\} \times \mathbb{C}^2_{x_3,x_4}$$

Q: primitive collection

$\{\rho_1, \rho_2\}$ $\{\rho_3, \rho_4\}$

$$\overset{(*)}{\Rightarrow} Z(\Sigma) = V(x_1, x_2) \cup V(x_3, x_4) = \mathbb{C}^2_{x_3,x_4} \cup \mathbb{C}^2_{x_1,x_2}$$

Now: $(\mathbb{C}^*)^{\Sigma(1)} \curvearrowright (\mathbb{C})^{\Sigma(1)} \setminus Z(\Sigma)$

$\cup\!\mid$

$G \curvearrowright \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$

$\downarrow$ $/G$ (want)

X_Σ

$\tilde{\sigma} \subseteq \mathbb{Z}^{\Sigma(1)} \otimes \mathbb{R}$ $\quad$ $\tilde{\sigma}$: cone gen. by e_ρ, $\rho \in \sigma(1)$

$\uparrow$ how?

$\sigma \subseteq N_{\mathbb{R}}$

$\tilde{\sigma} \subseteq \mathbb{R}^{\Sigma(1)}$

$\{\tilde{\sigma}\} \rightsquigarrow$ fan $\tilde{\Sigma}$ in $\mathbb{R}^{\Sigma(1)}$

Prop. $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$ is a toric var. associated to $\tilde{\Sigma}$.

Pf: Take $\tilde{\Sigma}_0$ to be the fan of $\mathbb{C}^{\Sigma(1)}$

$\Rightarrow \tilde{\Sigma}$ is a subfan of $\tilde{\Sigma}_0$.

$$\tilde{\Sigma}_0 \xleftrightarrow{\text{orb-cone}} \mathbb{C}^{\Sigma(1)}$$

$$\tilde{\Sigma} \longleftrightarrow \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$$

need: remove orbits in $\tilde{\Sigma}_0 \setminus \tilde{\Sigma}$

$\Updownarrow$ orb-cone.

remove min elements in $\tilde{\Sigma}_0 \setminus \tilde{\Sigma}$

$\Updownarrow$

primitive $C \subseteq \Sigma(1)$

$\Updownarrow$

removing $V(x_\rho,\ \rho \in C)$. ☐

$\exists$ map: $\mathbb{Z}^{\Sigma(1)} \to N$

$e_\rho \mapsto u_\rho$

$$1 \to \ker \to (\mathbb{C}^*)^{\Sigma(1)} \to T_N \to 1$$

$\ker = G$

toric morphism: $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \xrightarrow{\pi} X_\Sigma$

Thm (Cox). This π is a a GIT quotient, may be written as

$$X_\Sigma \simeq \mathbb{C}^{\Sigma(1)} /\!/ G$$

$$(\simeq (\mathbb{C}^{\Sigma(1)})^{ss} /\!/ G$$

$$\simeq (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)) /\!/ G).$$

Also, Σ simplicial

$\Updownarrow$

π is geometric.

(G-orb $\leftrightarrow$ pt in quotient) e.g.

$\mathbb{C}^4 \setminus Z(\Sigma)$

$Z(\Sigma) = \mathbb{C}^2_{x_1, x_2} \cup \mathbb{C}^2_{x_3, x_4}$

$(\mathbb{C}^*)^2 \curvearrowright \mathbb{C}^4$

linearization $(1, 1)$

$\Updownarrow$

$(\mu, \lambda) \cdot (x_1, x_2, x_3, x_4) = (\mu x_1, \mu x_2, \lambda x_3, \lambda x_4)$

$$\frac{\mathbb{C}^4 \setminus (\mathbb{C}^2_{x_1, x_2} \cup \mathbb{C}^2_{x_3, x_4})}{(\mathbb{C}^*)^2} = \mathbb{P}^1 \times \mathbb{P}^1$$

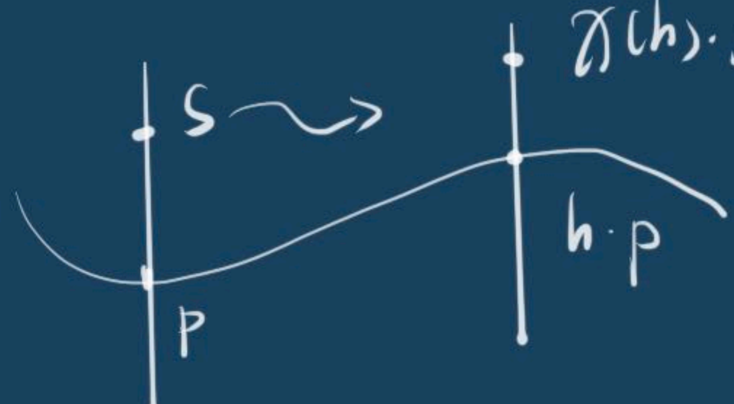
Note: this is a geometric quotient.

Generalizations:

$\mathbb{C}^n$: as a $T = (\mathbb{C}^*)^n$ - toric var.

$(\mathbb{C}^*)^k = H \subseteq T$, fix $\chi \in M_T$
$\quad\ \ \rotatebox{90}{$\in$}$
$\quad\ \ h$

$\rightsquigarrow$ L_χ l.b. H- equv. by $\chi|_H$.



play two games :

1). $\mathbb{C}^n /\!/_\chi H$ is a toric var.
$\quad$ GIT quotient.

2). take symplectic quotient.
$\rightsquigarrow$ hypertoric var.

What does 2) mean ?

Step 1): moment map

$\mu : T^* \mathbb{C}^n \to \mathrm{Lie}(H)^* \ (\cong \mathbb{C}^k)$

restrict to $\mathbb{R}$ if necessary.

Step 2): take :

$$\mu^{-1}(0) /\!/_\chi H = \mu^{-1}(0)^{\chi\text{-ss}} /\!/ H$$

Step 1) : $T^*\mathbb{C}^n \to \mathbb{C}^k$ $\qquad H \overset{\eta}{\hookrightarrow} T$

$$(x, w) \mapsto \eta^*\left(\sum_{i=1}^n (x_i w_i) e_i\right)$$

$x \in \mathbb{C}^n$

w: fiber coord.

(for $\mathbb{R}$: $\eta^*\left(\frac{1}{2} \sum_{i=1}^n (|x_i|^2 - |w_i|^2) e_i\right)$

Step 2) : $H \curvearrowright \mu^{-1}(0)$

$\nu : H \times \mu^{-1}(0) \to \mu^{-1}(0)$

$\rightsquigarrow \nu^* : \mathcal{O}_{\mu^{-1}(0)} \to \mathcal{O}_{H \times \mu^{-1}(0)}$

$$\mu^{-1}(0) /\!/_\chi H = \mathrm{Proj} \bigoplus_{m \geq 0} H^0(\mathcal{O}_{\mu^{-1}(0)})^{\chi^m}$$

$$= \mathrm{Proj} \bigoplus_{m \geq 0} \{ f \in H^0(\mathcal{O}_{\mu^{-1}(0)}) \mid \nu^*(f) = \chi^m \otimes f \}$$

$\mu^{-1}(0)/\!/_\chi H$ is called a hypertoric var.

The structure of the hypertoric var is encoded in some hyperplane arrangement.

how?

$$0 \to H \xrightarrow{\eta} T \to T/H \to 0 \qquad \chi : \text{char. of } H$$

$$0 \to \mathfrak{h} \xrightarrow{\eta} \mathfrak{t} \to \mathfrak{t}/\mathfrak{h} \to 0 \qquad \eta^*(\chi) \in \mathfrak{t}^*$$

$\mathfrak{h} \cong \mathbb{C}^k$, $\mathfrak{t} \cong \mathbb{C}^n$, $\mathfrak{t}/\mathfrak{h} \cong \mathbb{C}^d$; $\quad \eta^*(\chi) = t = (t_1, \dots, t_n)$

choose n $\mathbb{Z}$-vectors $a_i \in (\mathfrak{t}/\mathfrak{h})$, $i = 1, \dots, n$

$$H_i = \{ v \in (\mathfrak{t}/\mathfrak{h})^* \mid v \cdot a_i + t_i = 0 \}, \quad i = 1, \dots, n$$

$\mathcal{A} = \{H_1, \dots, H_n\}$ ⇝ determines χ

⇝ $HT(\mathcal{A}) := \mu^{-1}(0)/\!/_\chi H$

Now: $\mathcal{A}$: $\{H_i, i = 1, \dots, n\}$ w/ $t_i = 0$.
(central hyperplane arrangement)

$\tilde{\mathcal{A}}$: $\{\tilde{H}_i = \{H_i + t_i = 0\}\}$, t generic

⇝ $\pi : HT(\tilde{\mathcal{A}}) \to HT(\mathcal{A}) = \mathrm{Spec}\, \mathcal{O}_{\mu^{-1}(0)}^{H} \ni 0$

$\pi^{-1}(0) =: \mathcal{L}(\tilde{\mathcal{A}})$ core of $\tilde{\mathcal{A}}$

Thm. $\mathcal{L}(\tilde{\mathcal{A}})$ is a (seminormal) union of toric var's which corresp. to bounded region of the hyp. arr.

e.g.

$\mathcal{L}(\tilde{\mathcal{A}}) = \mathbb{P}^2 \cup Bl_1 \mathbb{P}^2$ glued along $\mathbb{P}^1 \subseteq \mathbb{P}^2$, $E \subseteq Bl_1 \mathbb{P}^2$

$\mathcal{L}(\tilde{\mathcal{A}}) = \mathbb{P}^2 \cup \mathbb{P}^2$ glued along a pt.