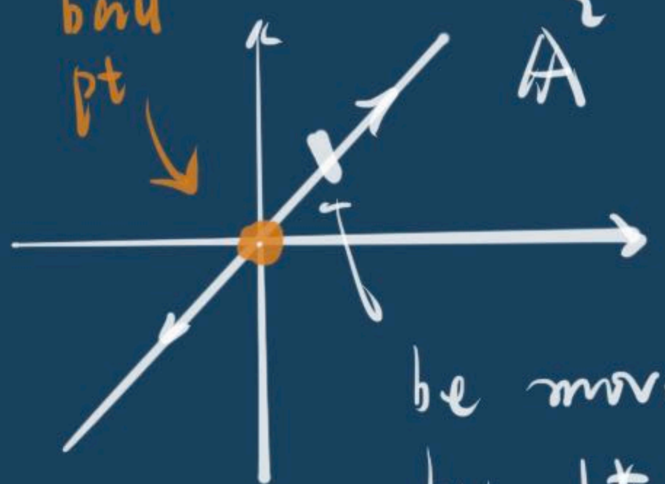
Geometric Invariant Theory
(general and in toric case).

Warm up: Q: how to parametrize lines through $(0,0)$ in $\mathbb{A}^2_{x,y}$?

A: $\mathbb{P}^1 = (\mathbb{A}^2 \setminus \{(0,0)\}) / \mathbb{C}^*$

bad pt



be moved by $k^* = \mathbb{G}_m(k)$

What did we do?

1). remove bad pts

2). take a quotient

$\Rightarrow$ get a parametrizing space (para. what?)

In general. no w/ 1) the quotient will be ugly, i.e. it might not be alg.

"correct obj" means (some) closed orbits in 2).

e.g. $G = \mathbb{G}_m(k) \curvearrowright \mathbb{A}^2_{x,y}$ via.

a). $\lambda(x,y) = (\lambda x, \lambda^{-1} y)$
$wt(x) = 1$, $wt(y) = -1$

b) $\lambda(x,y) = (\lambda x, \lambda y)$
$wt(x) = wt(y) = 1$

For a): ring of invariants:

alg: $k[x,y]^G = \{ f \in k[x,y] \mid g \cdot f = f, \forall g \in G \}$
$\quad = k[xy]$

since $\cdot\ \lambda(xy) = \lambda x \cdot \lambda^{-1} y = xy$

$\cdot$ only inv. monomials are $(xy)^k$

(need Poincaré series)

geo: $\operatorname{Spec} k[xy] = \mathbb{A}^1_c = \mathbb{A}^2_{x,y} / \mathbb{G}_m$, $c = xy$

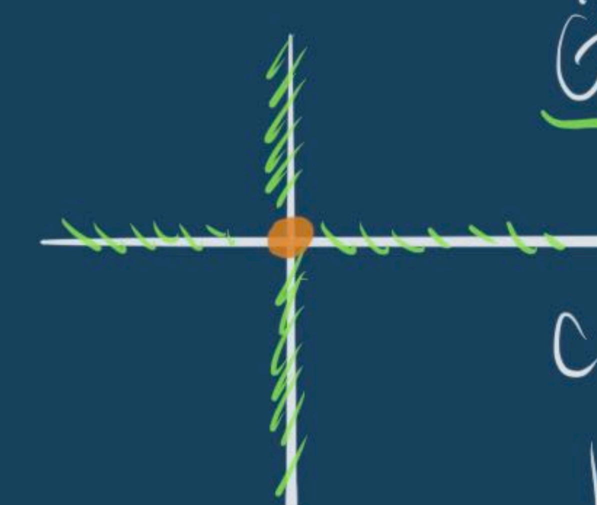
pts in the quotient:

$\mathbb{A}^1_c \ni c \neq 0 \Leftrightarrow xy = c$ which is a closed orb.

$c = 0 \Leftrightarrow xy = 0$

$\mathbb{G}_m \cdot (1,0) \amalg \mathbb{G}_m \cdot (0,1) \amalg \mathbb{G}_m \cdot (0,0)$

$\quad \mathcal{O}_{(1,0)} \qquad \mathcal{O}_{(0,1)} \qquad \mathcal{O}_{(0,0)}$



observation: $c = 0 \Leftrightarrow 3$ orbits but only one of them is closed.

b). ring of inv:

alg: $k[x,y]^G = k$

geo: $\operatorname{Spec} k[x,y]^G = pt = \mathbb{A}^2_{x,y}/G_m$

G_m-orb: $\begin{cases} y = cx \\ \underline{x = 0} \end{cases}$

Now: remove $\{x=0\}$ (geo)

add $\frac{1}{x}$ (alg)

$k[x, \frac{1}{x}, y]^G = k[\frac{y}{x}]$

$\Rightarrow (\mathbb{A}^2_{x,y} - \mathbb{A}^1_y)/G_m = \mathbb{A}^1_{\frac{y}{x}}$

X/G is larger even if X is smaller.

Main definitions:

$G \curvearrowright X$

Def 1: A <u>categorical quotient</u> Y is a var. s.t.

1). $G \curvearrowright Y$ trivially

2) $\exists$ G-eqv. morphism $f: X \to Y$

i.e. f commutes w/ the G-action.

3). universal property:

for another such pair $(Y', f': X \to Y')$

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Y \arrow[rr, dashed, "\exists!"'] & & Y' \end{tikzcd}$$

Def 2: A <u>geometric quotient</u> is a var Y satisfying 1) 2) above plus

- (set) pts in $Y \overset{1:1}{\longleftrightarrow}$ G-orbits in X.
- (topology) $\underset{\text{open}}{U \subseteq Y} \iff \underset{\text{open}}{f^{-1}(U) \subseteq X}$
- (sheaf) $\forall$ open $U \subseteq Y$, $\Gamma(U, \mathcal{O}_Y) \cong \Gamma(f^{-1}(U), \mathcal{O}_X)^G$

Rmk: if a geo. quotient $\exists$ $\Rightarrow$ it must be a cat. quot.

e.g. $G_m \curvearrowright \mathbb{A}^1_x$, $\lambda \cdot x = \lambda x$

no geo quotient, but the cat. quotient $= \operatorname{Spec} k$.

Rmk: in general

$G \curvearrowright R$: f.g. k-alg.

R^G is not necessarily f.g.

need: reductivity of G

e.g. reductive gps: GL, SL, Sp, $\mathbb{G}_m$, ...

from now on, G is always assumed to be reductive.

Main thms:

Affine case

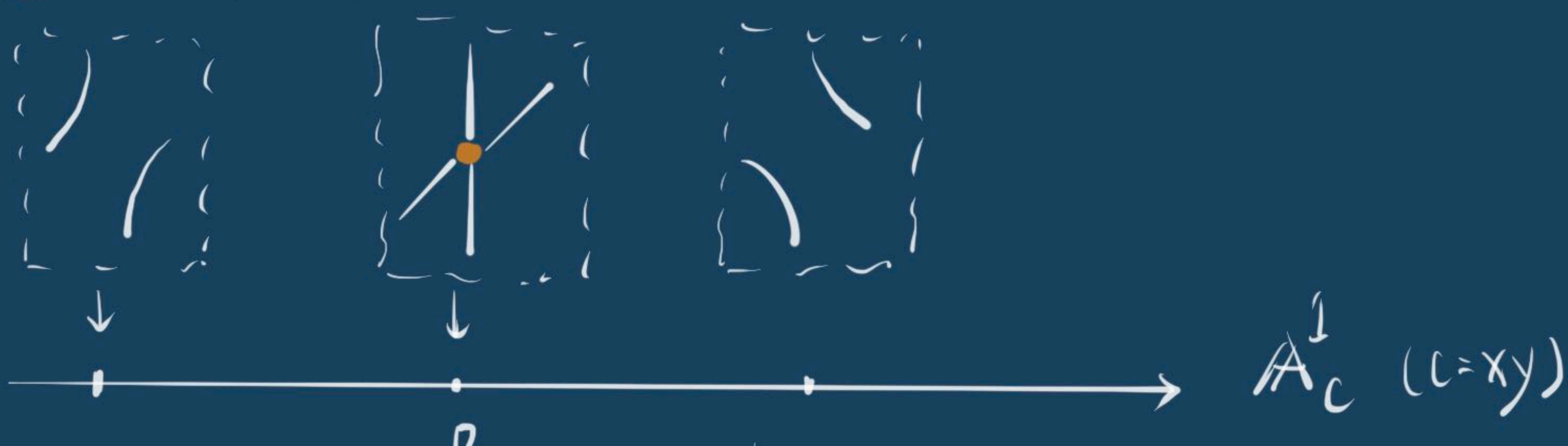
Thm (Nagata, Mumford)

- $G \curvearrowright X = \operatorname{Spec} R \Rightarrow \exists$ cat. quotient which is exactly $Y = \operatorname{Spec}(R^G)$
- pts in $Y \xleftrightarrow{1:1}$ orbit (closure) eqv. classes

$$G\cdot x_1 \sim G\cdot x_2 \Leftrightarrow \overline{G\cdot x_1}\cap \overline{G\cdot x_2} \neq \emptyset.$$

- in each orb. eqv class, $\exists!$ closed orb. $\subset$ $\forall$ other G-orb closure in the same class.

e.g. (Picture) $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ $wt(x)=1$, $wt(y)=-1$



Note: away from $0\in\mathbb{A}^1_c$, fibers are closed orb i.e. the quotient is geo.

Proj case.

(X, L) X: proj var
L: ample line bundle.

$$\begin{array}{c} G \curvearrowright L \\ \downarrow \\ G \curvearrowright X \end{array} \Leftrightarrow \text{alg}: G \curvearrowright R(X,L) = \bigoplus_{d\geq 0} H^0(X, L^{\otimes d})$$

$\tilde{X} = \operatorname{Spec} R(X,L)$ affine cone $\xrightarrow{\text{quotient}}$ $\tilde{Y} = \operatorname{Spec} R(X,L)^G$

$\downarrow$

$X = \operatorname{Proj} R(X,L)$ $\longrightarrow$ $Y = \operatorname{Proj} R(X,L)^G$

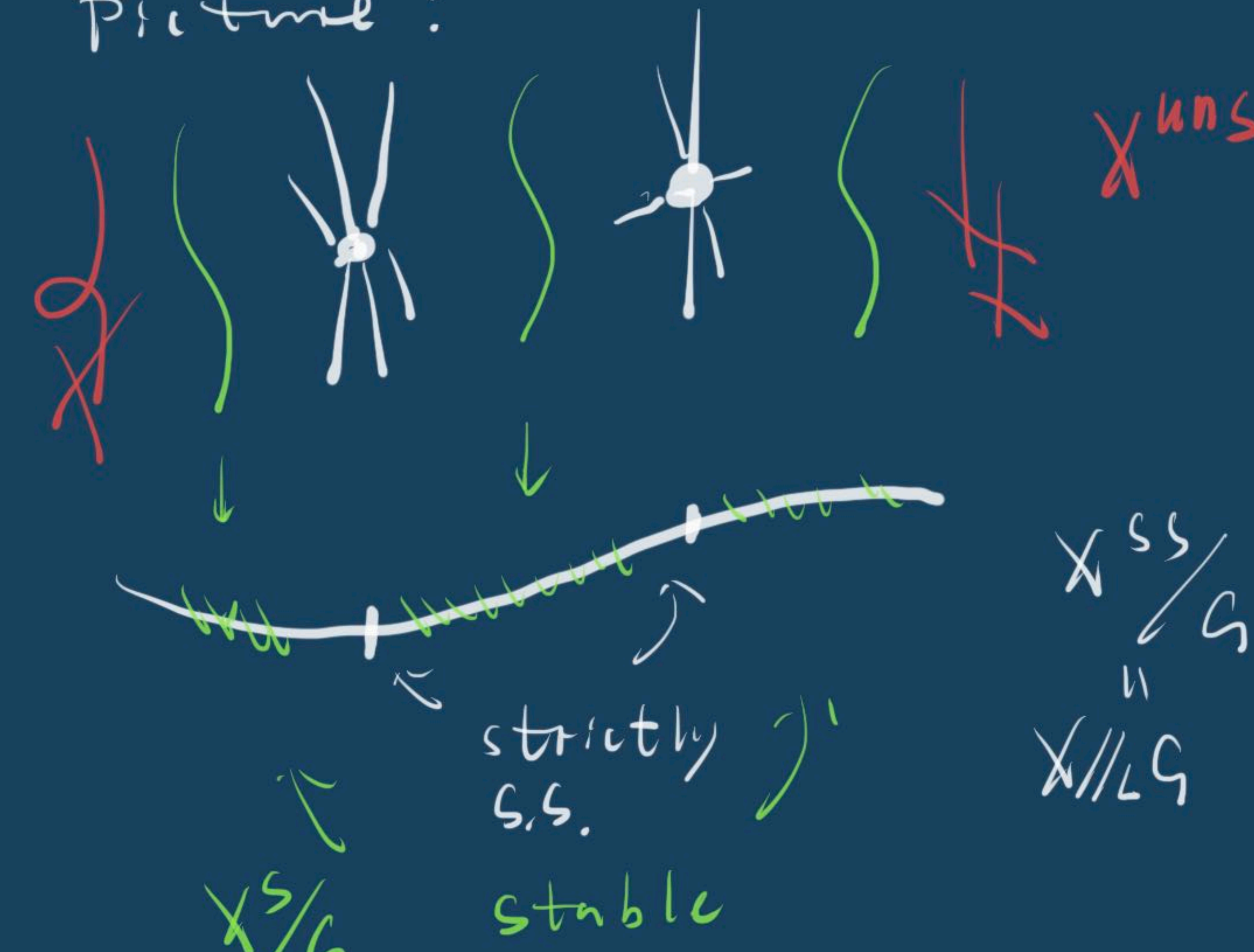
a pt on $\tilde{Y} \Rightarrow \exists!$ pt on Y except for the vertex

need: remove G-orb in $\tilde{Y}$ w/ $\overline{G\cdot\tilde{x}} \ni 0$. (i.e. orb ~ 0).

Def: $x \in X$ is

- unstable if $\overline{G\cdot \tilde{x}} \ni 0$ for a lift $\tilde{x}$. (X^{uns})
- semistable if x is not unstable. (X^{ss})
- stable if x is s.s. plus (X^{s})
  - a). $G\cdot x$ is closed
  - b). $G_x = \mathrm{Stab}_G(x)$ is finite.

Picture:



Thm (Mumford) $G \curvearrowright L$ ample
$\downarrow$
$G \curvearrowright X$ proj var. then

- $X^s \underset{\text{open}}{\subset} X^{ss} \underset{\text{open}}{\subset} X$
- X^{ss}/G is a cat. quotient, which is exactly $\mathrm{Proj}\, R(X,L)^G$
  $\shortparallel$
  $X/\!/_L G$ is called the GIT quotient (in proj case)

  pts in $X^{ss}/G \leftrightarrow$ G-orb. equ. classes. $\exists!$ closed orbit in each equ. class. $\subset$ any other G-orb closure in the same class
- X^s/G is a geometric quotient.
  $\Rightarrow$ pt $\leftrightarrow$ G-orb in X^s

e.g. $G_m \curvearrowright \mathbb{P}^1_{x,y} = X$, $L = \mathcal{O}(1)$

a) $\lambda[x:y] = [\lambda x : \lambda^{-1} y]$ b) $\lambda[x:y] = [\lambda x : \lambda y]$

a): unstable locus:
$0 = \mathbb{C}xy$, in $\mathbb{A}^2_{x,y}$: $\{x=0\} \cup \{y=0\}$
in $\mathbb{P}^1_{x,y}$: $[0:1]$ and $[1:0]$
So $X^{ss} = \mathbb{P}^1 \setminus \{0\} \cup \{\infty\} = k^*$
$X/\!/_L G = k^*/k^* = pt$

b): in $\mathbb{A}^2_{x,y}$ $\quad 0 \in \overline{G\cdot p}$
$\Rightarrow X^{ss} = \emptyset$, $X/\!/_L G = \emptyset$.

Rmk: Here we detect the unstable pts by hand. ↗ to find (Q: How?)

But in general, this is hard.

A:

Hilbert–Mumford numerical criterion:

Def: G: reductive alg. gp. A 1-parameter subgp (1-PS) is a nontrivial hom:

$$\lambda: \mathbb{G}_m \to G.$$

$$\begin{array}{l} G \curvearrowright V \\ \lambda \uparrow \\ \mathbb{G}_m \curvearrowright V \rightsquigarrow V = \bigoplus V_i, \quad V_i = \{v \in V \mid t\cdot v = t^i v\} \end{array}$$

weight.

· all wts $> 0 \iff \lim_{t\to 0} t^i x = 0$

$\iff 0 \in \overline{\mathbb{G}_m \cdot x}$

$\iff 0 \in \overline{G \cdot x}$

$\iff x$ is unstable

· all wts ≥ 0: $\iff$ strictly s.s.

· wts: only 0

$\lambda(t)\cdot x = x$

$\underline{\mathrm{Stab}_G(x) \supseteq \mathbb{G}_m}$ not finite $\Rightarrow$ not stable.

· wts: not only 0

$x = \sum_{i \geq 0} x_i \qquad x_i \in V_i$

non-closed orb $\Leftarrow \lim_{t\to 0} \lambda(t)\cdot x_i = 0$

Thm (HM) $G \curvearrowright V \ni x$

x is unstable $\iff \exists$ 1-PS s.t. x admits only $+$ or $-$ wts

s.s. $\iff$ " " all wts ≥ 0 (or ≤ 0)

s. $\iff \forall$ 1-PS $\exists$ both $\pm$ wts.

Transcript of GIT on moduli

Subject:	curve	vector bundles	quivers
grade:	A+	A+	A+
	surf	hyperplane arr	≥3-fold
	B	A+	D.

reason: HM is hard to compute.

e.g. 

n pts on $\mathbb{P}^1_{x,y}$

$\Updownarrow$

$H^0(\mathcal{O}_{\mathbb{P}^1}(n))$

$\ni$

$\sum a_i x^i y^{n-i}$

$\lambda: \mathbb{C}^*_t \to SL(2,\mathbb{C})$

$\lambda(t)\cdot[x:y] = [t^k x : t^{-k} y]$

$\lambda(t)\, x^i y^{n-i} = t^{ki} x^i\, t^{-k(n-i)} y^{n-i}$
$= t^{k(2i-n)} x^i y^{n-i}$

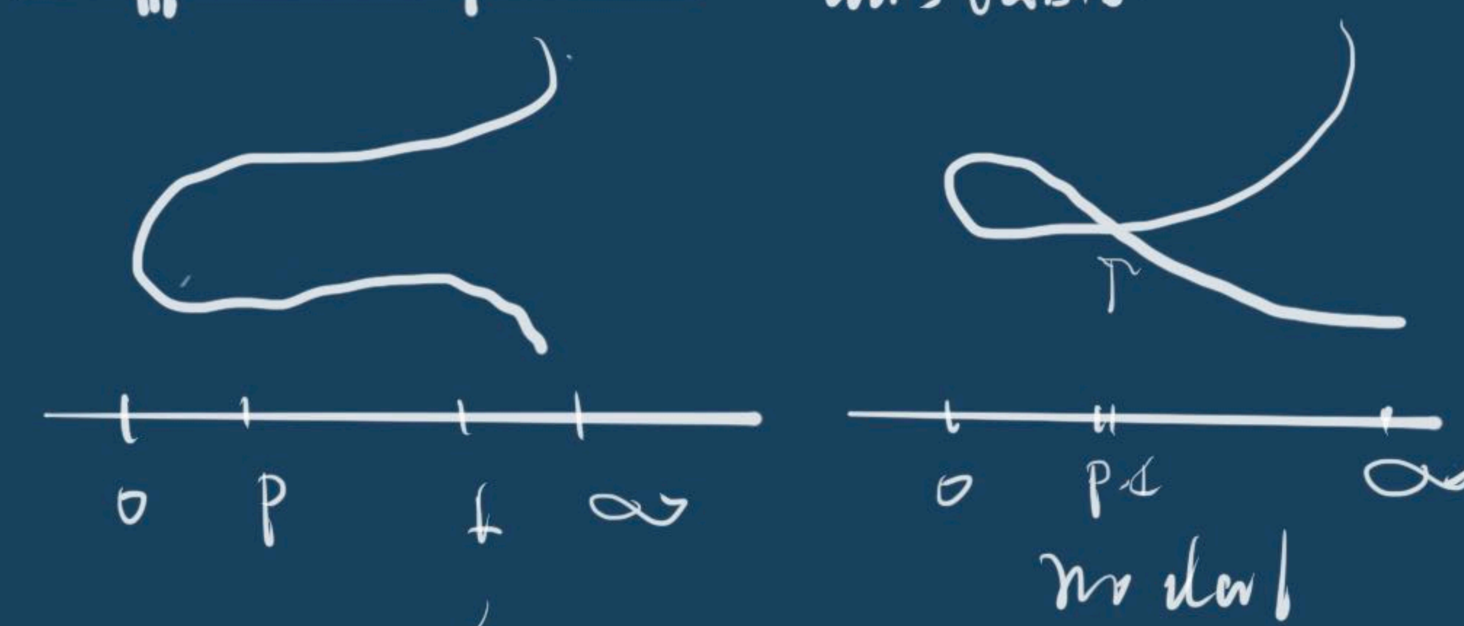
$x^i y^{n-i}$ has wt $k(2i-n)$

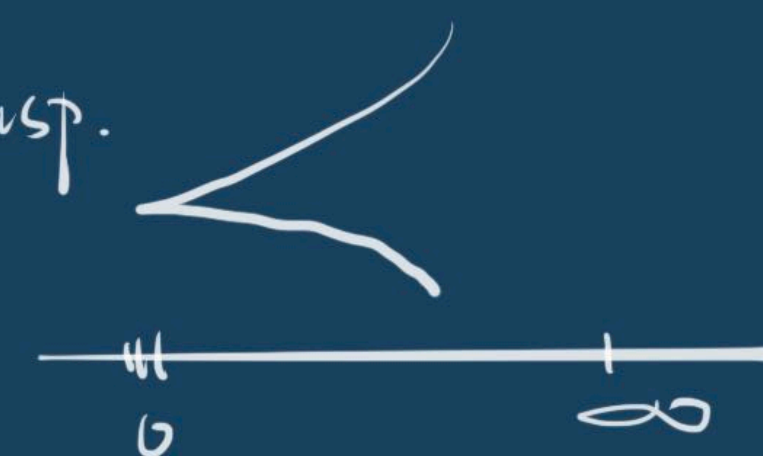
stable $\Leftrightarrow$ $\exists$ $2i > n$
(s.s.) $\quad 2i < n$
$(\geq / \leq)$

$\Leftrightarrow$ mult. of $x < n/2$
y $(\leq)$

$\Leftrightarrow$ at most $< n/2$ pts can be accumulated.
$(\leq n/2)$

4 pt on $\mathbb{P}^1$, in $\overline{M}_{0,4}$, we don't see unstable

Rmk: 

cusp. 

obs: $\underbrace{G \curvearrowright L}$ can be changed
$\downarrow$ $\quad\Updownarrow$
$\underbrace{G \curvearrowright X}$ fixed $\quad X \overset{\varphi_L}{\hookrightarrow} \underbrace{\mathbb{P}^N \curvearrowleft G}$

$X /\!\!/_L G$ is not canonical, it dep's on the linearization $G \curvearrowright L$.

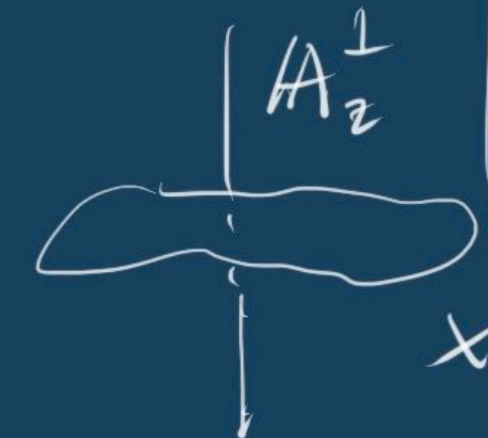
$\rightsquigarrow$ Variation of GIT.

easiest case: $G = \mathbb{G}_m = T$

$T \curvearrowright X = \operatorname{Spec} R$ (w/ a trivial l.b.)

$\Updownarrow$

$\mathbb{Z} \cong M$-grading of R, say $R = \bigoplus_{i \in \mathbb{Z}} R_i$



Goal: study $X/\!/T$

define a grading on $R[z]$: $R_i \subset R[z]_i$ (z: coord. on fiber $\mathbb{A}^1_z$.)

(dif from the deg grading by $z^{\boxed{i}}$). $z \in R[z]_{-n}$

$\Downarrow$

T-linearization on $\mathcal{O}_X$

quotient: $\operatorname{Proj} R[z]^T = \operatorname{Proj} R[z]_0$

$= \operatorname{Proj} \bigoplus_{i \in \mathbb{N}} R_{ni} z^i$

$n = 0$: $\operatorname{Proj} R_0[z] = \operatorname{Spec} R_0 =: X/\!/0$

$n > 0$: $X/\!/+$

$n < 0$: $X/\!/-$

For $X/\!/0$, the function field: $\{ r/s \mid r, s \in R_0 \}$

For $X/\!/+$, " " : $\{ r/s \mid r, s \in R_{di}$ for some $i \geq 0 \}$.

$k(X/\!/+) \hookleftarrow k(X/\!/0)$

$\frac{r}{s} \mapsto \frac{rt^i}{st^i}$ by choosing $0 \neq t \in R_{-d}$

$\Rightarrow X/\!/+ \overset{\text{bir}}{\dashrightarrow} X/\!/0$

Moreover: $R_0[z] \to \bigoplus R_{ni} z^i$

$\operatorname{Proj} R_0[z] \leftarrow \operatorname{Proj} \bigoplus R_{ni} z^i$

proj morphism.

$\Rightarrow$ Prop.

$$\begin{tikzcd} X/\!/- \arrow[rr, dashed, "\eta"] \arrow[dr, "\text{bir}"'] & & X/\!/+ \arrow[dl, "\text{bir}"] \\ & X/\!/0 & \end{tikzcd}$$

Rmk: Actually η is a flip.

~~toric~~ :

(X, L) polarized toric var $\longleftrightarrow$ lattice polytope $Q \subseteq M_{T,\mathbb{R}} = M_T \otimes_{\mathbb{Z}} \mathbb{R}$

$H \subset T$ subtorus   linearization $H \curvearrowright L$

$$\downarrow$$

$$H \curvearrowright X$$

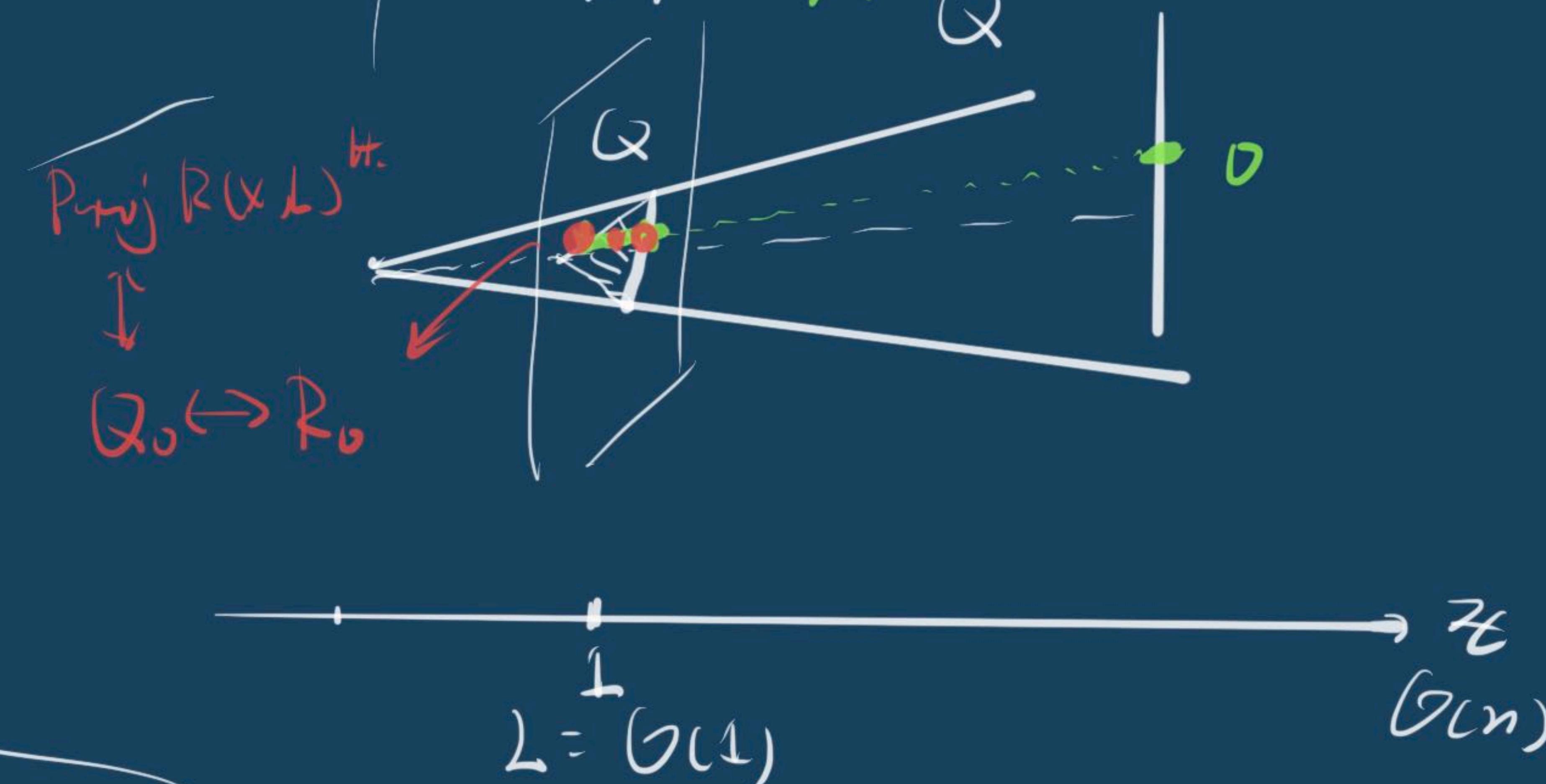
Goal: to find $X /\!/_L H = ?$

$X = \operatorname{Proj} R(X, L)$

$\updownarrow$

Q

$M_T \to M_H$ how?



Observation:

1). $(X, L) \longleftrightarrow Q$

$(X, L^{\otimes d}) \longleftrightarrow dQ$

$\operatorname{Proj} R(X, L) \cong \operatorname{Proj} R(X, L^{\otimes d}) \Rightarrow$ get the same GIT quotient

$\Rightarrow$ makes sense to talk about $\mathbb{Q}$-polarizations.

2). $H \curvearrowright R(X, L) \Leftrightarrow R(X, L)$ is graded by M_H (wt)

$\Rightarrow R(X, L)^H = R_0, \ 0 \in M_H$

$\Rightarrow X /\!/_L H = \boxed{\operatorname{Proj} R_0}$

3). $H \hookrightarrow T \Leftrightarrow \operatorname{Hom}(T, \mathbb{C}^*) \xrightarrow{\varphi} \operatorname{Hom}(H, \mathbb{C}^*)$

$$\begin{array}{ccc} \| & & \| \\ M_T & \to & M_H \\ \ker \varphi & \to & 0 \end{array}$$