Toric Fano Var's

Def: A sm. proj var X is <u>Fano</u> if: $-K_X$ ample.

in dim 2, they are called del Pezzo surfaces (dP)

e.g. $\mathbb{P}^n$, $-K_{\mathbb{P}^n}=(n+1)H$ ample ✓

$X=\prod_{i=1}^{k}\mathbb{P}^{n_i+1}$, $-K_X=\mathcal{O}(n_1+1,\dots,n_k+1)$ ample ✓

Thm. (in dim 2)

$$X \text{ is a sm dP} \iff \begin{cases} \mathbb{P}^1\times\mathbb{P}^1 \\ \text{or} \\ \mathrm{Bl}_{p_1,\dots,p_n}\mathbb{P}^2, \quad n\le 8 \end{cases}$$

where $p_1,\dots,p_n$ are pts on a general position.

i.e. no 3 pts on a line

no 6 pts on a conic

no 8 pts on a nodal cubic

The degree of the dP is

$$\deg(\mathbb{P}^1\times\mathbb{P}^1)=8$$

$$\deg(\mathrm{Bl}_{p_1,\dots,p_n}\mathbb{P}^2)=K_X^2=d=9-n$$

e.g. (higher dim).

$X_d\subseteq\mathbb{P}^n$ sm. hypersurf. by adjunction

$$K_{X_d}=(K_{\mathbb{P}^n}+X_d)\big|_{X_d} = -(n+1)H+dH = (-n-1+d)H$$

need: $-K_{X_d}$ ample, i.e. $d<n+1$.

Or more general:

$X_{d_1,\dots,d_k}=\bigcap_{i=1}^{k}X_{d_i}\subseteq\mathbb{P}^n$ complete intersection

adjunction (mult. times) $\Rightarrow$

$$K_X=\Big(-n-1+\sum_{i=1}^{k}d_i\Big)H$$

ample: need $\sum d_i<n+1$

e.g. in $\mathbb{P}^3$

$d=2$: $\mathbb{P}^1\times\mathbb{P}^1$  deg 8.

$d=3$. $X_3\subseteq\mathbb{P}^3$, cubic surf. is $(\mathrm{Bl}_{5\text{pts}}\mathbb{P}^1\times\mathbb{P}^1\cong)\ \mathrm{Bl}_{6\text{pts}}\mathbb{P}^2 \supseteq 27$ lines

Classification problem:
(up to deformation types)

Fano: Classified Fano 3-folds X w/ $\rho(X) = 1$.

Iskovskikh: redid this in a modern way.
found 1 or 2 cases Fano missed

Mukai - Umemura: found 1 case Fano, Isk missed

Isk: $b_2 = 2$. 17 types
Mukai-Mori: higher. 88 types.
done.

for Fano's w/ mild sing's. ε-lc.

know: finitely many deformation type
thm due to Birkar (BAB conjecture)

Def: A sing. proj var X is Fano if $-K_X$ is
- $\mathbb{Q}$-Cartier (mK_X Cartier) so called ($\mathbb{Q}$-) Gorenstein.
- ample

Rmk: if the sing's are log terminal X is called log Fano.

The toric case

Thm. A complete toric var $X = X_\Sigma$ is Fano $\Leftrightarrow$

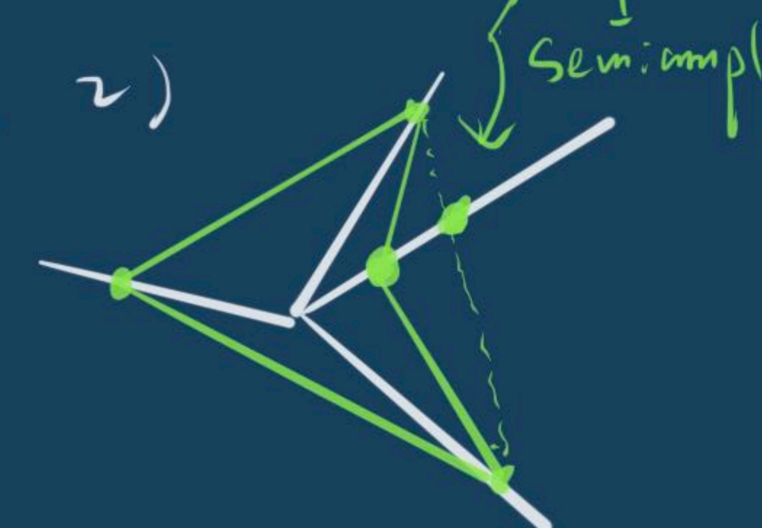
1) $\forall$ max cone σ, generators of $\sigma(1)$: $\{v_i\}_{i\in\sigma(1)}$ are on the same hyperplane.
2) $\{v_i\}_{i\in\Sigma(1)}$ are exact $\mathrm{Vert}(\mathrm{Conv}\{v_i\}_{i\in\Sigma(1)})$.

Rmk: violators:

of 1).

$-K_X$ is not $\mathbb{Q}$-Cartier

of 2)



$-K_X$ not ample

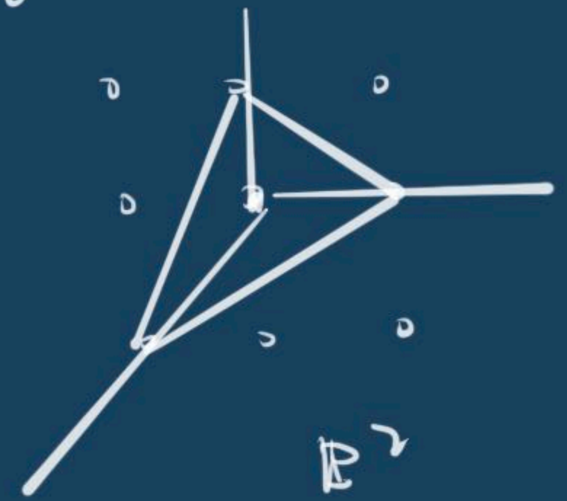
Def: $Q = \mathrm{Conv}\{v_i\}$, if 1) 2) are satisfied, Q is called reflexive.

equ: Q is reflexive iff
a) full dim
b) $Q = \{n \in N_{\mathbb{R}} \mid \langle n, u_F\rangle \geq -1, F \text{ facet}\}$

u_F: normal prim. vector of F in M

$Q \ni 0$ is the only interior $\mathbb{Z}$-pt $\Leftarrow$ b) facets have lattice dist. 1 from 0.

e.g. dim 2 $N_{\mathbb{R}} \simeq \mathbb{R}^2$



$\mathbb{P}^2$

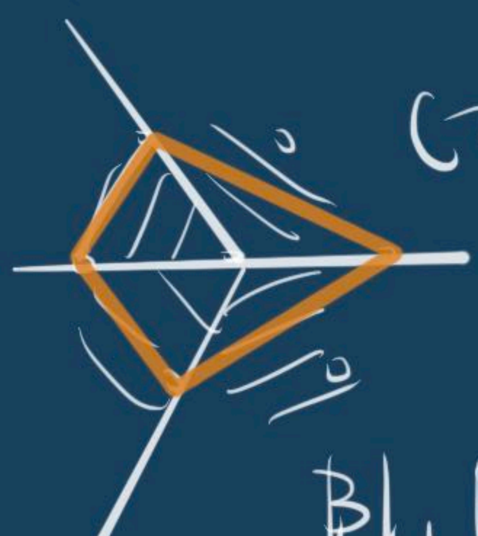
=

(1)

Prop/exercise:

There are only five sm. toric dP.

(2)

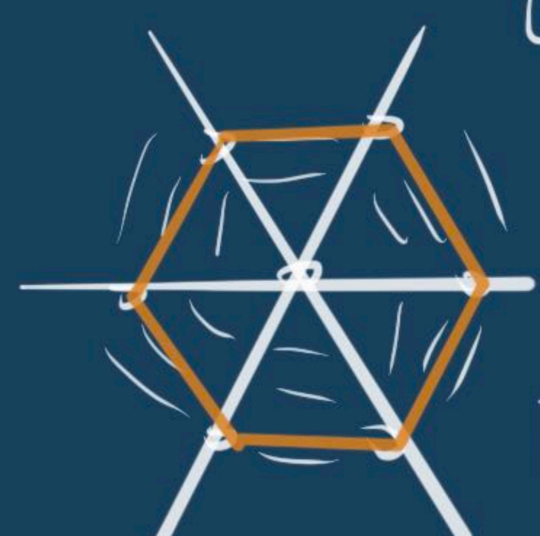


$Bl_1 \mathbb{P}^2$

(3)

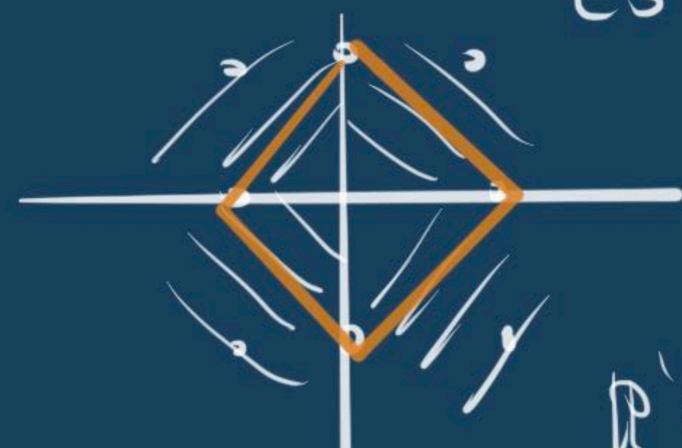
$Bl_2 \mathbb{P}^2$

(4)



$Bl_3 \mathbb{P}^2$

(5)



$\mathbb{P}^1 \times \mathbb{P}^1$

each of these gives a reflexive polytope.

Rmk/HW: if "sm" is removed

$\Rightarrow$ 16 polytopes reflexive.

(Topological / Combinatorical / Hodge theoretic) Mirror Symmetry

due to Batyrev.

Def: A complete var X^n of dim n is Calabi-Yau (CY)

if $\begin{cases} K_X \simeq \mathcal{O}_X \\ H^i(X, \mathcal{O}_X) = 0, \quad i = 1, \cdots, n-1 \end{cases}$

e.g. dim 1: ell. curves

dim 2: K3 surf's. Hodge diamond

$h^q(X, \Omega^p_X)$:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

dim 3: CY 3-folds

Hodge diamond X :

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & a & & 0 & \\ 1 & & b & & b & & 1 \\ & 0 & & a & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Slogan/goal of top. mirror sym: CY 3-folds appear in pairs (families)

$X \rightsquigarrow X^\vee$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & b & & 0 & \\ 1 & & a & & a & & 1 \\ & 0 & & b & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$\downarrow$

find such pairs as more as possible.

Baty rev: Toric Fano var's appear in pairs, say

how? → $Y \leftrightarrow Y^\vee$

$\cup \qquad \cup$ hypersurf's

$X \leftrightarrow X^\vee$

CY's we want.

Dual polytopes: polarized toric

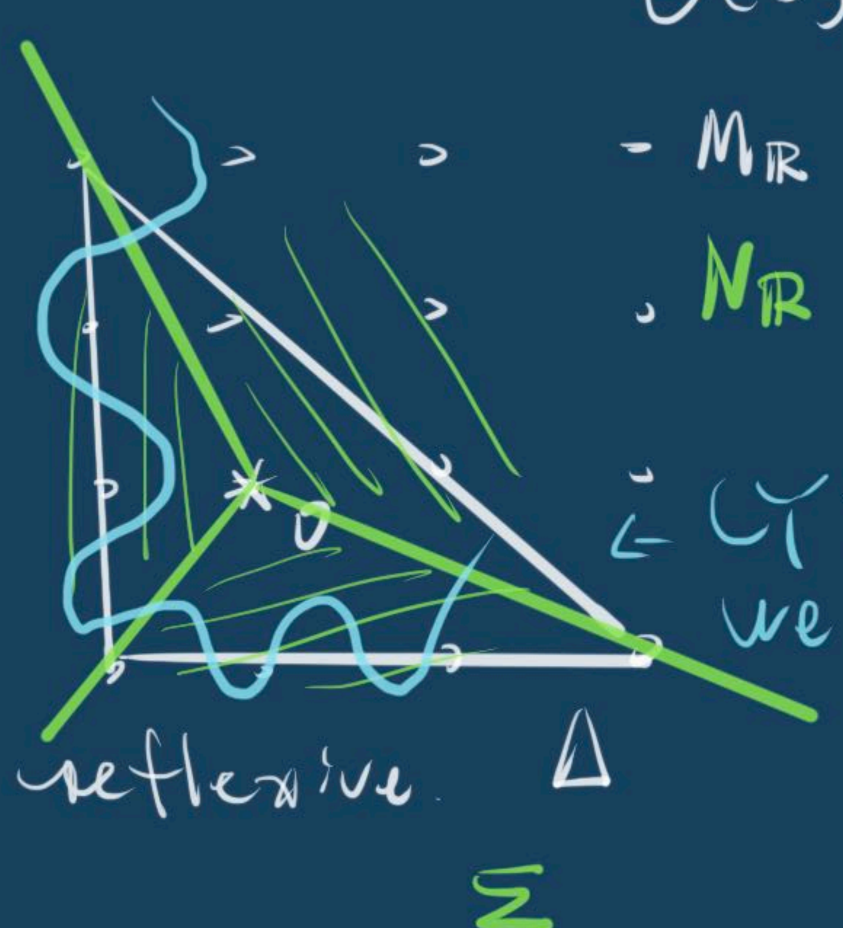
$\Delta \longleftrightarrow \Delta^\circ \to Y_\Delta \leftrightarrow Y_{\Delta^\circ} = Y^\vee$

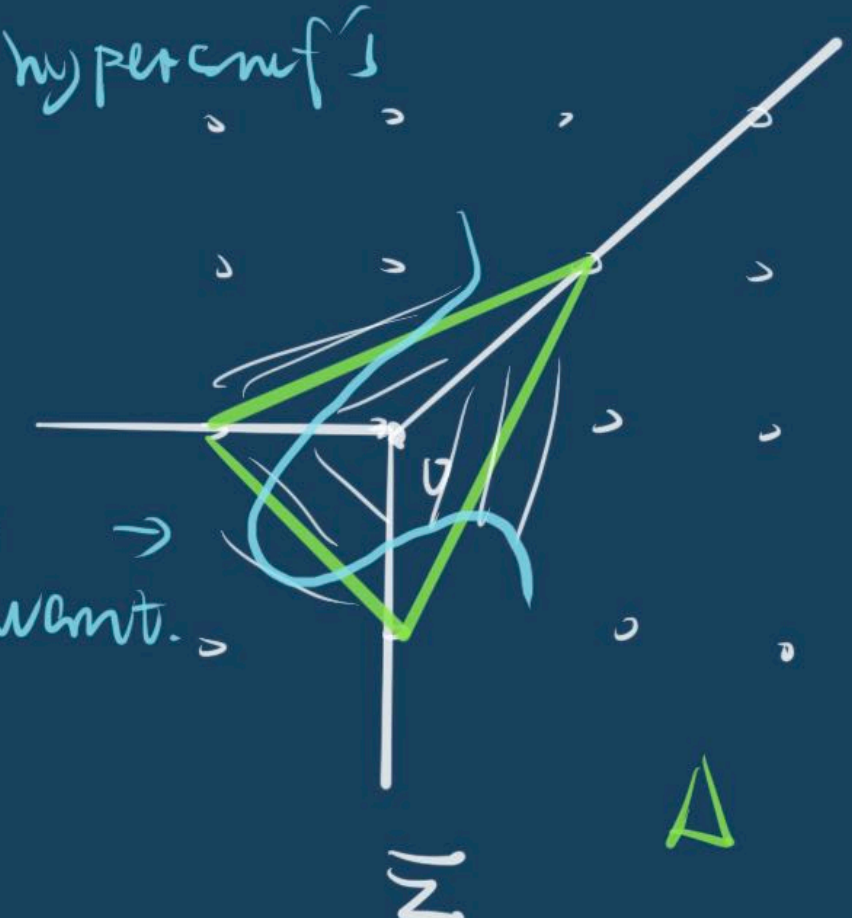
$\cap$ $M_\mathbb{R}$ how?

Want: X CY, need: $0 \sim K_X = \underline{(K_{Y_\Delta} + X)}|_X$

i.e. polarization is $-K_X$

e.g. $Y = (\mathbb{P}^2, -K_{\mathbb{P}^2} = 3H)$
$\mathcal{O}(3)$





In general:

Def: $P \subseteq M_\mathbb{R} \rightsquigarrow$ polar dual P°

$P^\circ := \{ n \in N_\mathbb{R} \mid \forall m \in P,\ -\langle m, n\rangle \leq 1 \}$

$(= \mathrm{Conv}\{u_F\}_{F \text{ facet}})$.

Rmk: facets of P $\longleftrightarrow$ vert. of P°

codim 2 .. $\longleftrightarrow$ edges ..

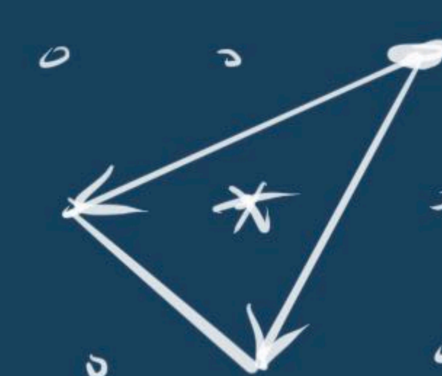
$\vdots \qquad \vdots$

vert .. $\longleftrightarrow$ facets ..

Lemma: P reflexive $\Rightarrow$ so is P°

$P^{\circ\circ} = P$.

Problem: e.g.



is sing.

need: resolve sing's

Recall: resolution of sing's

$\updownarrow$

toric: subdiv. of cones

e.g.

$\mathrm{Bl}_{\text{sing pt}} X_6$

$\updownarrow$

insert v

Fix $\Delta \subseteq M_{\mathbb{R}}$, $\Sigma \subseteq N_{\mathbb{R}}$
polytope, normal fan of Δ

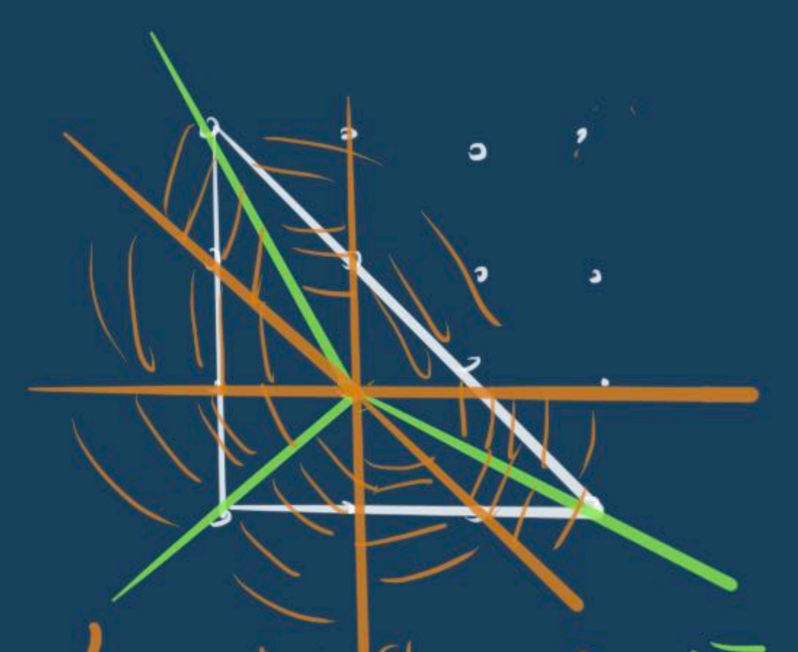
Def: A fan $\Sigma' \subseteq N_{\mathbb{R}}$ is a proj subdiv of Σ if

1) Σ' refines Σ
2) $\Sigma'(1) = \langle v_i \rangle_{i \in \Sigma'(1)}$, $v_i \in (\Delta^\circ \cap N) \setminus \{0\}$
3) $X_{\Sigma'}$ is proj and simplicial ($\mathbb{Q}$-factorial).

In 2), if $\{v_i\}_{i \in \Sigma'(1)} = \Delta^\circ \cap N \setminus \{0\}$, Σ' is called maximal. (MPS)

Rmk: MPS exists.

e.g.



Σ' : MPS of Σ.

What can we say for Σ'?

- $X_{\Sigma'}$ is Gorenstein
- $X_{\Sigma'} \xrightarrow{f} X_{\Sigma}$ is crepant. $K_{X_{\Sigma'}} = f^*(K_{X_{\Sigma}})$
- $-K_{X_{\Sigma'}}$ is semiample ($\Rightarrow$ nef)
- if MPS $\Rightarrow$ $X_{\Sigma'}$ has terminal sing'.

$$\begin{array}{ccc} \Delta : \text{refl. polytope} & \longrightarrow & \text{normal fan of } \Delta \\ \dim n & & \text{has a MPS } \Sigma \\ \updownarrow & & \updownarrow \\ \Delta^\circ & \rightsquigarrow & \Sigma^\circ \end{array}$$

$\Rightarrow$ family of hypersurf's which are CY, dim $n-1$ $\quad V \in |-K_{X_\Sigma}|$ $\updownarrow$ $V^\circ \in |-K_{X_{\Sigma^\circ}}|$

Thm (Batyrev)

$h^{1,1}(V) = h^{n-2,1}(V^\circ)$

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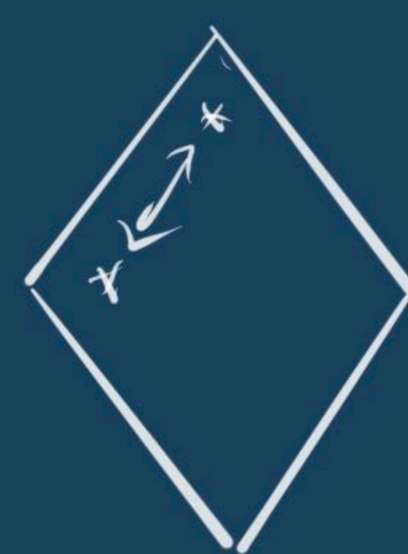


note: $H^{p,q}(V) = H^q(V, j_*\Omega^p_{V_{sm}})$

$j : V_{sm} \hookrightarrow V$.

Idea of pf:

Step 1: $H^{1,1}(V) \supseteq H^{1,1}_{toric}(V)$, $\quad H^{n-2,1}(V) \supseteq H^{n-2,1}_{poly}(V)$

Find: a) $H^{1,1}_{toric}(V)$, $H^{n-2,1}_{poly}(V)$

What are they?

a). $\{D_i\}$: T-inv. div's of X_Σ

$\rightsquigarrow \{D_i'\}$, $D_i' = D_i \cap V$

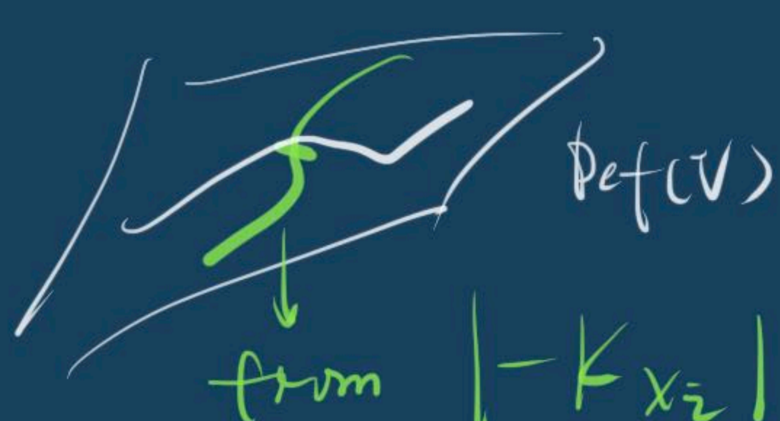
↳ gen. a subspace in $H^{1,1}(V)$ $\overset{!!}{=}$ $H^{1,1}_{toric}(V)$

b). $H^{n-2,1}(V) \hookrightarrow H^1(V, T_V)$

V: Gorenstein terminal $\Rightarrow$ $H^1(V, T_V)$ is still the deformation space

$|-K_{X_\Sigma}| \rightsquigarrow$ a subsp. in $H^1(V, T_V)$

$\| \quad H^{n-2,1}_{poly}(V)$



Step 2: Computation:

a). $h^{1,1}_{toric}(V) = l(\Delta^\circ) - n - 1 - \sum_{\Gamma^\circ} l^*(\Gamma^\circ)$

b). $h^{n-2,1}_{poly}(V) = l(\Delta) - n - 1 - \sum_{\Gamma} l^*(\Gamma)$

where $l(Q) = \#$ $\mathbb{Z}$-pt on Q

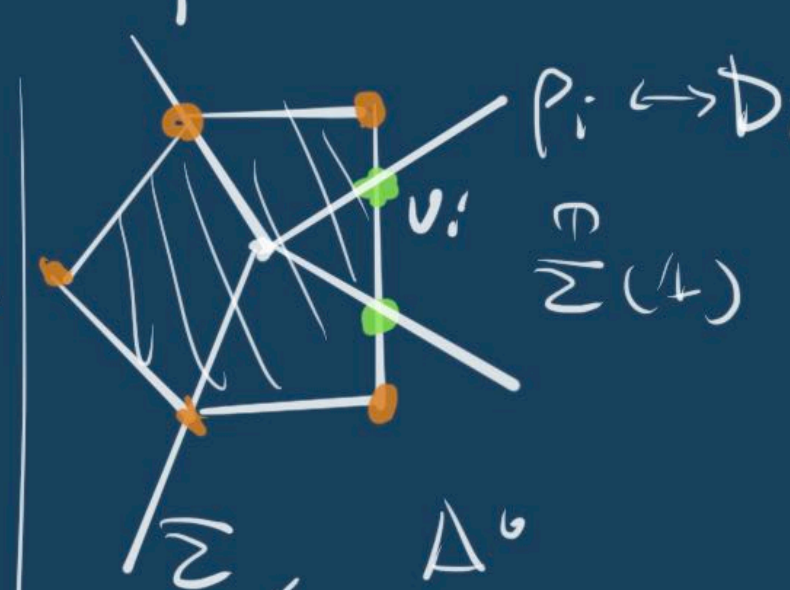
$l^*(Q) = \#$ $\mathbb{Z}$-pt on Q, not on any facet.

Γ / Γ° runs over all facets of Δ / Δ°

Note: a) b) $\Rightarrow$ $h^{1,1}_{toric}(V) = h^{n-2,1}_{poly}(V^\circ)$

$h^{n-2,1}_{poly}(V) = h^{1,1}_{toric}(V^\circ)$

explain a):



$X_\Sigma \xrightarrow{f} X_\Delta$ is a toric bl. up.

$f(D_i) = pt$ for green rays.

$\Sigma(1)' = \{v_i \mid v_i$ not in the interior of any facet$\}$

a general $V \cap D_i = \emptyset$ $\quad f(D_i) = pt$

$\ni$

$|-K_{X_\Sigma}|$

$\Rightarrow H^{1,1}_{toric}(V) = \langle D_i' \mid \rho_i \in \Sigma(1)' \rangle$

↑ not free.

Q: What is the relation?

A: given by

$$\begin{array}{ccccccccc} 0 & \to & M & \to & \mathbb{Z}^{\Sigma(1)'} & \to & \mathbb{Z}^{\Sigma(1)'}/M & \to & 0 \\ & & & & & & \downarrow \subseteq \otimes \mathbb{C} & & \\ & & & & & & H^{1,1}(V) & & \end{array}$$

$\Rightarrow$ dim: $h^{1,1}(V) = |\Sigma(1)'| - \dim(M_{\mathbb{R}})$

$$= \underbrace{l(\Delta^\circ)}_{\text{all } \mathbb{Z}\text{pts}} - \underbrace{1}_{0} - \underbrace{\textstyle\sum_{\Gamma^\circ} l^*(\Gamma^\circ)}_{\text{green pts.}} - n$$

explain b):

Lemma: $\dim(\mathrm{Aut}(X_\Sigma)) = n + \sum_\Gamma l^*(\Gamma)$

Q: Who is in $|-K_{X_\Sigma}|$?

A: $V = \overline{Z_f} \subseteq X_\Sigma$

$\cup$

$Z_f = V(f) \subseteq T_N$ where f has monomial on Δ.

para. f.

$\rightsquigarrow \dim \; l(\Delta)$

f, cf ↘ ↙ the same V.

param. space of $\{f\}$ has dim $l(\Delta)-1$.

kill Aut. by the lemma.

$\Rightarrow h^{n-2,1}_{poly}(V) = l(\Delta) - 1 - n - \sum_\Gamma l^*(\Gamma)$

Now: we are good for "mirror" on toric v.s. poly.

Step 3: non toric part

a) $h^{1,1}(V) - h^{1,1}_{toric}(V) = \sum_{\theta^\circ} l^*(\theta^\circ)\cdot l^*(\hat{\theta}^\circ)$

b) $h^{n-2,1}(V) - h^{n-2,1}_{poly}(V) = \sum_\theta l^*(\theta)\, l^*(\hat{\theta})$

where θ/θ° runs over all codim 2 faces of Δ/Δ°, $\hat{\theta}/\hat{\theta}^\circ$ is the dual face of θ/θ° in Δ°/Δ

a) $L = R$

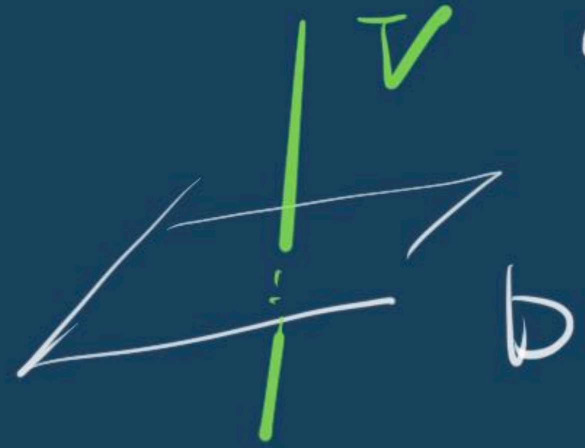
b) $L = R$

Explain a): $L = R$

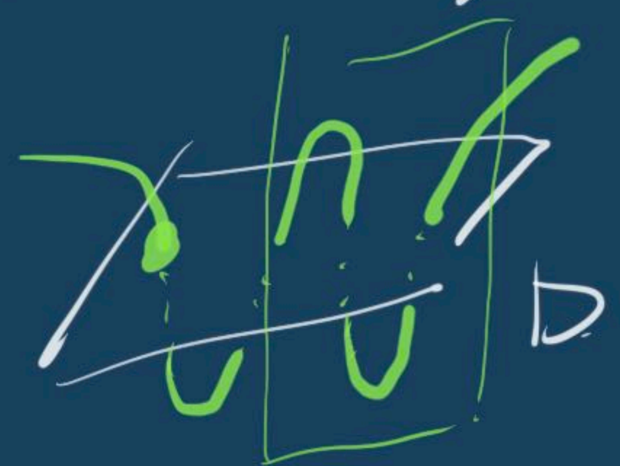
What do we miss on L?

we counted: D toric div.

all $D|_V$ as below:



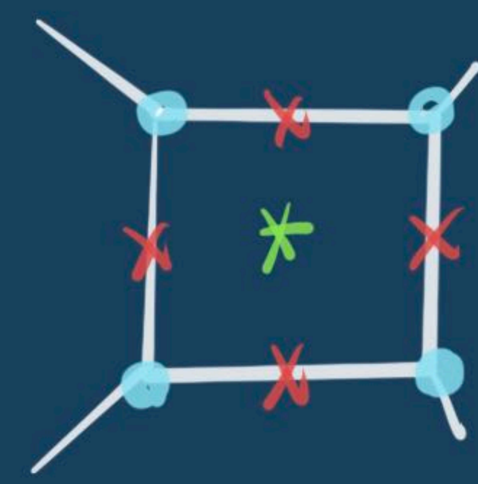
But actually, may exist:



may have reducible intersection $V \cap D$.

$$\begin{array}{ccc} f: X_\Sigma & \longrightarrow & X_\Delta \\ \cup & & \\ V & \longrightarrow & f(V) \end{array}$$

Previous:



face θ^0

$\updownarrow$

$\hat{\theta}^0$

$*$'s: can by ignored.

o's: counted correctly.

×'s: missed.

$\dim \theta^0 + \dim \hat{\theta}^0 = n - 1$

codim $\theta^0 \geq 3$ can be ignored.

$\Rightarrow \dim \hat{\theta}^0 \geq 2$. $\Uparrow$

Bertini $\Rightarrow X_{\hat{\theta}^0} \cap f(V)$ irr.

Now: codim $\theta^0 = 2 \Rightarrow \dim \hat{\theta}^0 = 1$. curve.

Compute: $\# \{ f(V) \cap X_{\hat{\theta}^0} \} = f(V) \cdot X_{\hat{\theta}^0}$

$= \deg X_{\hat{\theta}^0}$

$= \mathrm{Vol}(X_{\hat{\theta}^0})$

$= \ell^*(\hat{\theta}^0) + 1$

each interior pt in θ^0

contribute: $\ell^*(\hat{\theta}^0)$

irr. cpts

in total:

add $\ell^*(\theta^0) \cdot \ell^*(\hat{\theta}^0)$.

b) Gysin spectral seq.