Divisors on toric var's.

General story: X var / scheme $/ k = \bar{k}$ equi. dim, finite type

Weil divisors: $\sum n_i D_i$, $n_i \in \mathbb{Z}$ (or $\mathbb{Q}$ by $\otimes \mathbb{Q}$)

$\underline{WDiv(X)}$ D_i : irr. codim 1 subvar. $\subset X$

effective if $n_i \geq 0$, all i.

Cartier divisors: locally principal (defined by a single eqn).

$\underline{CDiv(X)}$ a global section of $\mathcal{K}^*/\mathcal{O}^*$

$\mathcal{K}^*$: sheaf of invertible rational functions

$\mathcal{O}^*$: " " regular " "

i.e. Cartier data:

$X = \bigcup \mathcal{U}_\alpha$ (open cover), $f_\alpha \in \mathcal{K}^*(\mathcal{U}_\alpha)$ s.t. over $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$

$f_\alpha / f_\beta \in \mathcal{O}^*(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$

$\exists$ homomorphism:

$\nu: CDiv(X) \to WDiv(X)$

on $\mathcal{U}_\alpha$ $f_\alpha \mapsto div(f_\alpha)$ ← divisor of zeros & poles

ν is injective if X is normal.

$\rightsquigarrow$ Class gp: $Cl(X) = WDiv(X)/\text{lin}$

$CaCl(X) = CDiv(X)/\text{lin}$

$WDiv(X)$: $D \overset{\text{lin}}{\sim} 0$ if $\exists f$, $D = div(f)$

$CDiv(X)$: $D' \overset{\text{lin}}{\sim} 0$ if $\exists f$, $f_\alpha = f$, $\forall \alpha$

s.e.s.:

$0 \to \mathcal{O}^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}^* \to 0$

$\rightsquigarrow$ l.e.s.

$H^0(\mathcal{K}^*) \to H^0(\mathcal{K}^*/\mathcal{O}^*) \to$

$\hookrightarrow \underline{H^1(\mathcal{O}^*)} \to H^1(\mathcal{K}^*) \to H^1(\mathcal{K}^*/\mathcal{O}^*) \to$

$\hookrightarrow \cdots$

Thm. X var / Noe. scheme reduced.

$\Rightarrow H^1(\mathcal{O}^*) = CaCl(X)$

$\qquad \| $

$\quad Pic(X)$

On toric var's :

$\Sigma \subset N_{\mathbb{R}}$

$X = X_\Sigma = \coprod \mathcal{O}_\sigma$

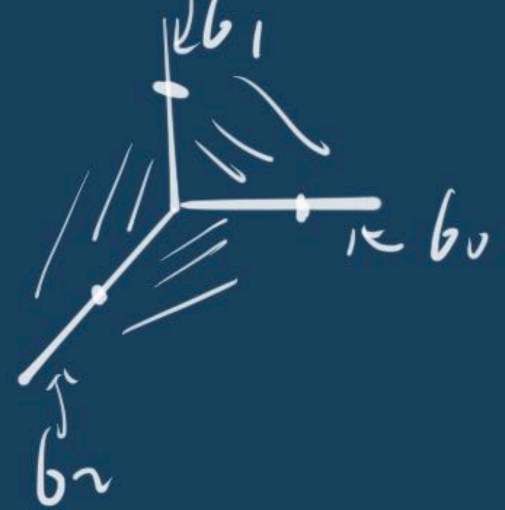
$= \bigcup \overline{\mathcal{O}_\sigma} = \bigcup V(\sigma)$

$V(\sigma) = \overline{\mathcal{O}_\sigma}$ is irr. toric

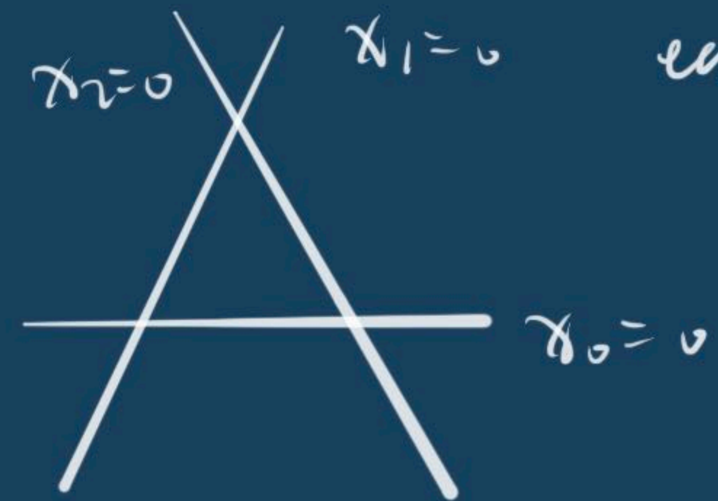
$\dim V(\sigma) = \operatorname{codim} \sigma$.

So for $\sigma \in \Sigma(1)$, $\{V(\sigma)\}$ are divisors, w/ cpts in $X \setminus T_N$.

e.g.



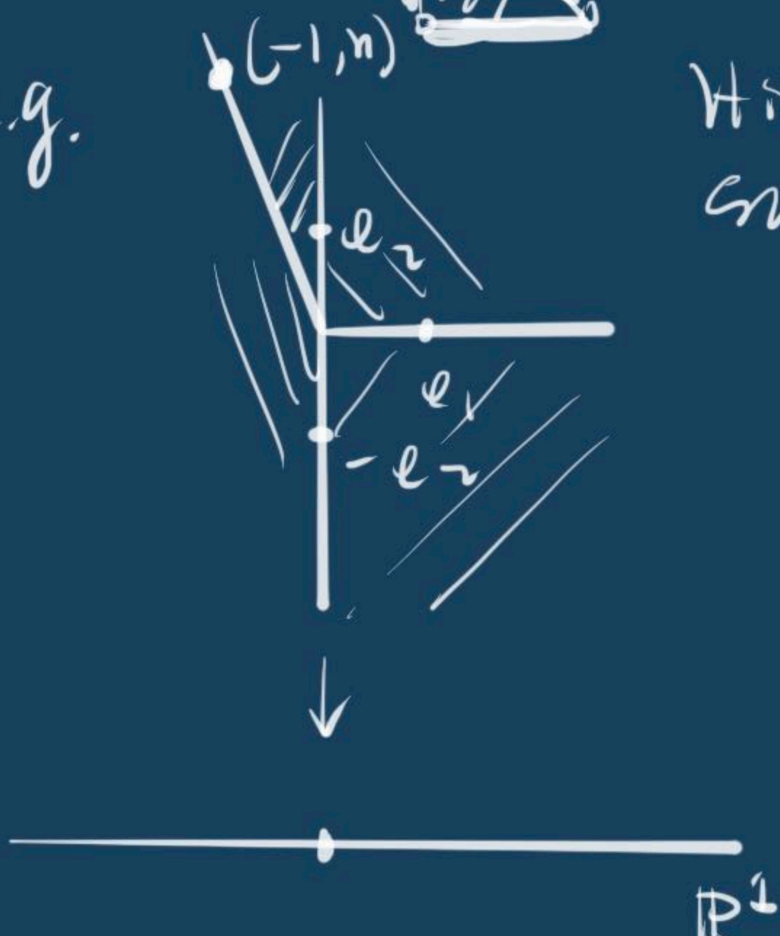
$X_\Sigma = \mathbb{P}^2$



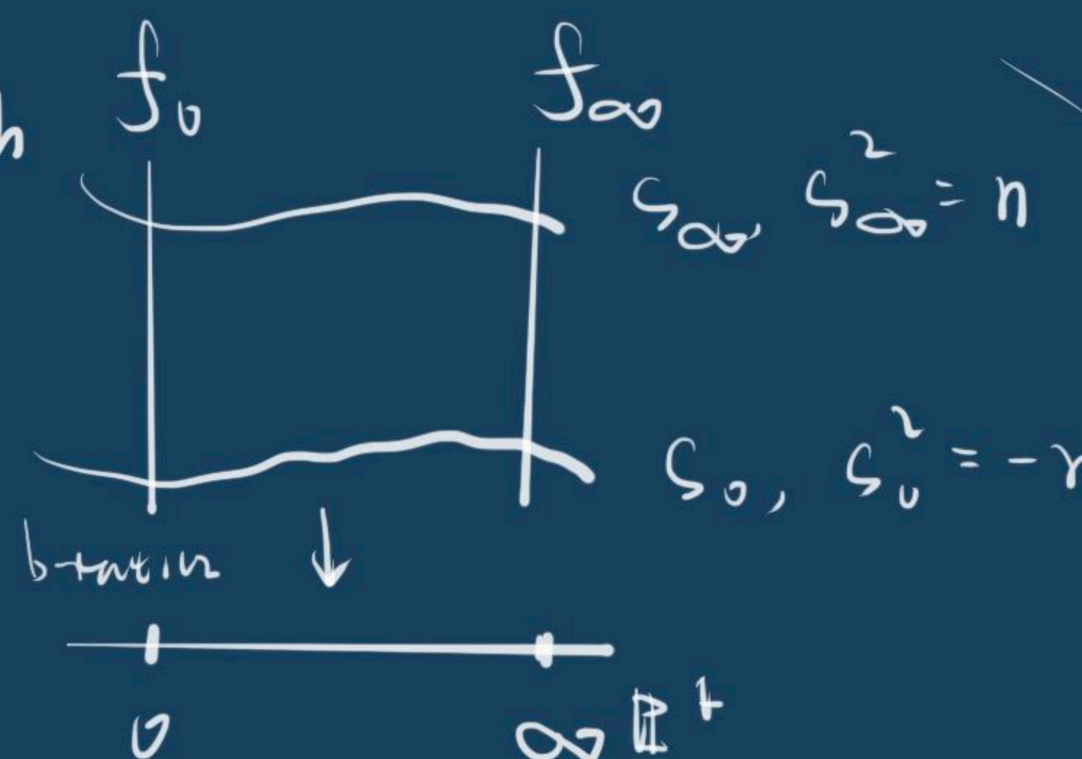
each $\{x_i=0\}$
$\|$
$\mathbb{P}^1$

dual picture: say $(\mathbb{P}^2, \mathcal{O}(1))$

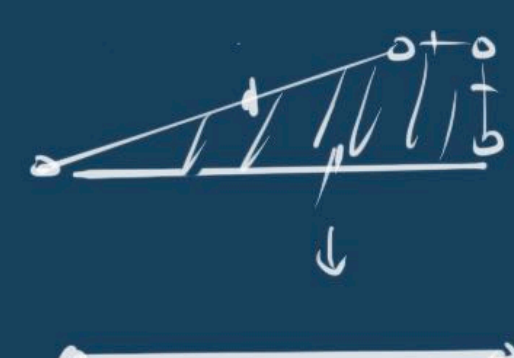
e.g.



Hirzebruch surf. $\mathbb{F}_n$



dual:



in $M_\mathbb{R}$: Q

e.g. $\mathbb{P}^2$

$$\begin{array}{ccccccc} 0 \to & M & \to & \mathbb{Z}^{\Sigma(1)} & \to & \mathrm{Pic}(\mathbb{P}^2) & \to 0 \\ & \wr\| & & \wr\| & & \wr\| & \\ & \mathbb{Z}^2 & & \mathbb{Z}^3 & & \mathbb{Z} & \end{array}$$

X normal $\Rightarrow$

s.e.s. ①: $M \to T_N\text{-}W\mathrm{Div}(X) \to \mathbb{Z}^{\Sigma(1)}/\mathrm{im} \to 0$

$$\begin{array}{ccccccc} & & \downarrow & & \| \text{①} & & \\ 0 \to & P\mathrm{Div}(X) & \to & W\mathrm{Div}(X) & \to & \mathrm{Cl}(X) & \to 0 \\ & \| & & \uparrow & & \uparrow & \\ 0 \to & P\mathrm{Div}(X) & \to & C\mathrm{Div}(X) & \to & \mathrm{CaCl}(X) & \to 0 \\ & & & \uparrow & & \| \text{②} & \end{array}$$

s.e.s ②: $M \to PL(\Sigma) \to PL(\Sigma)/\mathrm{im} \to 0$

$M \to T_N\text{-}W\mathrm{Div}(X) = \mathbb{Z}^{\Sigma(1)}$

$m \mapsto \sum_i (m, v_i) D_i$, v_i: $\mathbb{Z}$- generators of σ_i

D_i: div $V(\sigma_i)$

Def: $PL(\Sigma)$

A piecewise linear function on Σ is: a continuous $\mathbb{R}$- valued function φ on $|\Sigma|$ s.t.

- $\varphi|_\sigma$ is linear, $\forall \sigma \in \Sigma$.
- $\varphi|_\sigma$ is $\mathbb{Z}$-valued on $\sigma \cap N$.

i.e. $\exists\, m_\sigma \in M$, $\varphi|_\sigma(x) = (m_\sigma, x)$

Thm: ① ② "=" s.e.s. ① ②.

Q: When do we have $M \hookrightarrow \mathbb{Z}^{\Sigma(1)}$?

A: $M \hookrightarrow \mathbb{Z}^{\Sigma(1)} \Leftrightarrow M \hookrightarrow PL(\Sigma) \Leftrightarrow |\Sigma| = N_\mathbb{R}$

the fan must be complete

Cor (of the thm).

X sm $\Leftrightarrow$ $\left(\begin{matrix} \mathrm{WDiv}(X) \\ \| \\ \mathrm{CDiv}(X) \end{matrix}\right) \Rightarrow \mathrm{Pic}(X) = \mathrm{Cl}(X) = \mathbb{Z}^{\rho}$

$\rho = \#\Sigma(1) - \dim X$

picard rank.

Rmk: $M \to PL(\Sigma)$

$(km, -) \equiv 0 \Leftrightarrow k(m, -) \equiv 0$

$\Leftrightarrow (m, -) \equiv 0$

$\varphi_{km} = 0 \Leftrightarrow \varphi_m = 0$

means $\mathrm{coker}(M \to PL(\Sigma))$ is torsion free

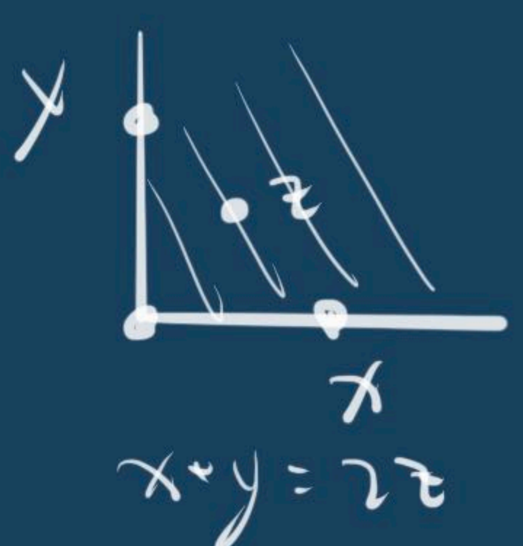
$\| $

$\mathrm{Pic}(X)$ $(= \mathrm{CaCl}(X))$

Be careful $\mathrm{Cl}(X)$ is not necessarily torsion free.

: D: not Cartier (D Cartier $\Rightarrow$ {sing. on X $\Rightarrow$ sing. on D})

$2D$: is Cartier



$M_{\mathbb{R}}$

$$\begin{matrix} M & \to & \mathbb{Z}^{\Sigma(1)} & \to & \mathrm{Cl}(X) & \to 0 \\ \| & & \| & & \wr\| & \\ \mathbb{Z}^2 & & \mathbb{Z}^2 & & \mathbb{Z}/2\mathbb{Z} & \end{matrix}$$

not torsion free.

Pf (of the thm) ① s.e.s. ①.

i.e. $M \to T_N\text{-WDiv}(X) \underset{\boxed{?}}{\to} \mathrm{Cl}(X) \to 0$

exact.

Recall: $D_1, \dots, D_r$, are irr. prime div's on X, $X \setminus \cup D_i = U$

$\rightsquigarrow$ exact seq: $\bigoplus_{j=1}^{r} \mathbb{Z} D_j \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$

taking div class taking restr.

For $X = X_\Sigma$, $\{D_\rho\}$ the set of T_N-WDiv's.

one-one corresp. $\Rightarrow$ $X \setminus \bigcup_{\rho \in \Sigma(1)} D_\rho = T_N$

$\Rightarrow$ $T_N\text{-WDiv}(X) \to \mathrm{Cl}(X) \to \underline{\mathrm{Cl}(T_N)} \to 0$

The coord. ring of T_N is

$k[M] = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

which is a UFD

$\Downarrow$

$\mathrm{Cl}(T_N) = 0$

So $\boxed{?}$ is true.

s.e.s. ①: $T_N\text{-WDiv}(X) \to \mathrm{Cl}(X)$

$D \mapsto 0$

$\Leftrightarrow D$ principal ($= \mathrm{div}(f)$)

$\mathrm{Supp}(D) \cap T_N = \phi \Rightarrow \mathrm{div}(f) = 0$ on T_N

$\Rightarrow f \in k[T_N]^* = k[M]^*$

$\Rightarrow f = c \cdot \chi^m$

i.e. $D = \mathrm{div}(\chi^m) = \sum_\rho (m, v_\rho) D_\rho$.

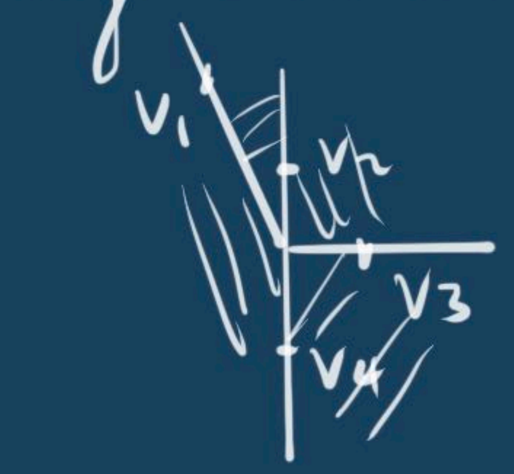
Rmk: $T\text{-}WDiv(X) \twoheadrightarrow Cl(X)$

this means $\forall D \in WDiv(X)$

$\exists D' \in T_N\text{-}WDiv(X)$

s.t. $D \overset{lin}{\sim} D'$

e.g. Hirzebruch surf. revisited



$$\begin{array}{lcl} v_1 = -e_1 + n e_2 & & D_1 \\ v_2 = e_2 & & D_2 \\ v_3 = e_1 & \longleftrightarrow & D_3 \\ v_4 = -e_2 & & D_4 \end{array}$$

$\{D_i\}_{i=1}^4$ generate $Cl(\mathbb{F}_n)$

relations: a) $0 \overset{lin}{\sim} div(\chi^{e_1}) = \sum_{i=1}^4 (e_1, v_i) D_i$

$= (e_1, v_1) D_1 + (e_1, v_3) D_3$

$= (1,0)\cdot(-1,n) D_1 + (1,0)\cdot(1,0) D_3$

$= -D_1 + D_3$

b) $0 \overset{lin}{\sim} div(\chi^{e_2}) = \sum_{i=1}^4 (e_2, v_i) D_i$

$= (e_2, v_1) D_1 + (e_2, v_2) D_2 + (e_2, v_4) D_4$

$= (0,1)\cdot(-1,n) D_1 + (0,1)(0,1) D_2 + (0,1)(0,-1) D_4$

$= n D_1 + D_2 - D_4$

By a) b):

$D_3 \overset{lin}{\sim} D_1$, $D_4 \overset{lin}{\sim} nD_1 + D_2$

$\Rightarrow Cl(\mathbb{F}_n)$ is free of rank 2, w/ generators $\{D_1, D_2\}$.

Q: When is a Weil divisor Cartier?

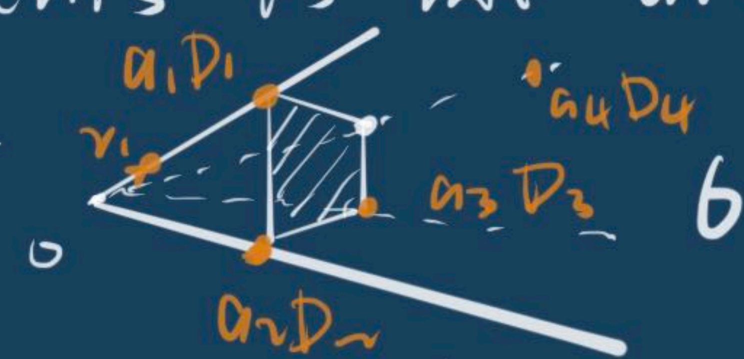
A: $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, $a_\rho \in \mathbb{Z}$

D is Cartier if for each $\sigma \in \Sigma(n)$

$\boxed{\exists m_\sigma}$ s.t. $\langle m_\sigma, v_\rho \rangle = -a_\rho$, $\forall \rho \in \sigma(1)$

Note: this is not always true

e.g.



Why? locally $\sigma \subseteq N_{\mathbb{R}} \rightsquigarrow X = U_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$

D: T_N-WDiv on X

$\rightsquigarrow \mathcal{O}_X(D) := \mathcal{O}_X(D)(U) = \left\{ \begin{array}{l} f \in \mathcal{K}(X) \\ div(f) + D \geq 0 \\ \text{on } U \end{array} \right\}$

is an inv. sheaf $\subseteq \mathcal{K}$

sheaf of total quot.

$\rightsquigarrow H^0(X, \mathcal{O}_X(D)) \subseteq H^0(X, \mathcal{K})$

- is M-graded
- is locally free rank 1

$\Rightarrow$ gen by one homog. elmt.

say $\chi^{-m} \Rightarrow D = div(\chi^m)$.

Def: If K_X is ($\mathbb{Q}$-)Cartier, then X is called ($\mathbb{Q}$-) Gorenstein.

Def: If $\forall$ Weil div D has a mult. mD Cartier, then X is called $\mathbb{Q}$-factorial.

$\Updownarrow$

cones are all simplicial.

e.g.

cannot happen

Canonical divisors.

Def: On X, a can div is $\mathrm{div}(\omega)$, ω = rational differential of deg = $\dim X$.

e.g. on $X = \mathbb{P}^1$, $X = \mathbb{A}^1_x \cup \mathbb{A}^1_y$, $y = \frac{1}{x}$ on $\mathbb{A}^1_x \cap \mathbb{A}^1_y$

pick dx, $dx = d\left(\frac{1}{y}\right) = \left(-\frac{1}{y^2}\right) dy$

$\mathrm{div}\left(-\frac{1}{y^2}\right) = -2 p_\infty$

in general $K_{\mathbb{P}^n} = -(n+1)H$.

$(\omega = \mathcal{O}_X(K_X))$

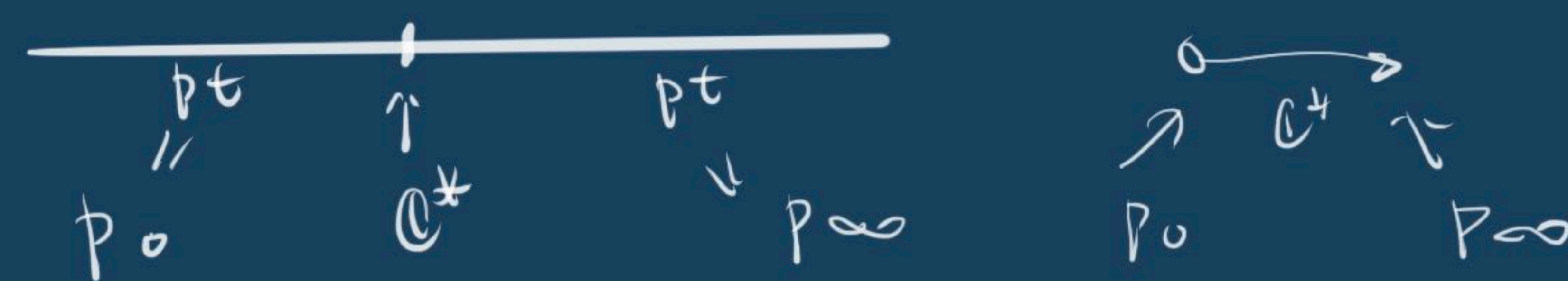
Problem: dx is not T-invariant.

Instead: look at $\frac{dx}{x}$

e.g. $\omega' = \frac{dx}{x} = \frac{1}{\left(\frac{1}{y}\right)} d\left(\frac{1}{y}\right) = y \cdot \left(-\frac{1}{y^2}\right) dy$

$= -\frac{dy}{y}$

$\mathrm{div}(\omega') = -p_0 - p_\infty$



In general:

Thm. $X = X_\Sigma \Rightarrow K_X = -\sum_{\rho \in \Sigma(1)} D_\rho$

e.g.

$K_{\mathbb{P}^2} = -D_0 - D_1 - D_2$

on $\mathbb{P}^2$: $D_i \overset{\text{lin}}{\sim} H$

So $K_{\mathbb{P}^2} = -3H$

$\mathbb{P}^n$, $e_1, \dots, e_n, -\sum_{i=1}^n e_i \rightsquigarrow K_{\mathbb{P}^n} = -(n+1)H$

Why? (the thm)

if X sm

As in Hartshorne:

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

In toric case, similarly:

$$0 \to \Omega^1_X \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_X(-D_\rho) \to \mathrm{Pic}(X) \otimes \mathcal{O}_X \to 0$$

$$\boxed{\begin{array}{c} \mathcal{O}_X^{r-n} \\ r = |\Sigma(1)| \end{array}}$$

By using the total Chern class:

$$1 \cdot c(\Omega^1_X) = c\left(\bigoplus_\rho \mathcal{O}_X(-D_\rho)\right)$$

cup prod

$$= \prod_\rho c(\mathcal{O}_X(-D_\rho)) = \underline{\prod_\rho (1 - [D_\rho])}$$

So $c_1(\omega_X) = c_1(\Lambda^n \Omega^1_X) = c_1(\Omega^1_X) = -\sum_\rho [D_\rho]$

$$\Rightarrow K_X = -\sum_{\rho \in \Sigma(1)} D_\rho$$

Numerical properties of toric div's / line b's.

X: normal proj (just complete)

$$\left\{ \begin{array}{c} \mathbb{Q}\text{-Cartier} \\ \text{div's} \end{array} \right\} \times \{\text{curves}\} \to \mathbb{Q}$$

$$D \cdot C$$

$$\{\text{line bundles}\} \times \{\text{curves}\} \to \mathbb{Z}$$

$$L \cdot C = \deg(L|_C)$$

Def: D is nef if $D \cdot C \geq 0$ $\forall$ effective C.

Rmk: Ample $\Rightarrow$ nef

$\not\Leftarrow$ not true.

$Bl_1 \mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$

f^*H is nef, not ample

$$(N^1(X) \otimes \mathbb{R}) \times (N_1(X) \otimes \mathbb{R}) \to \mathbb{R}$$

$$D \cdot C$$

$N^1 : \{\text{div's}\}/\equiv$

Thm (Kleiman)

D ample $\Leftrightarrow$ $D \cdot C > 0$ for all C in the closure the cone of curves $\overline{NE(X)}$

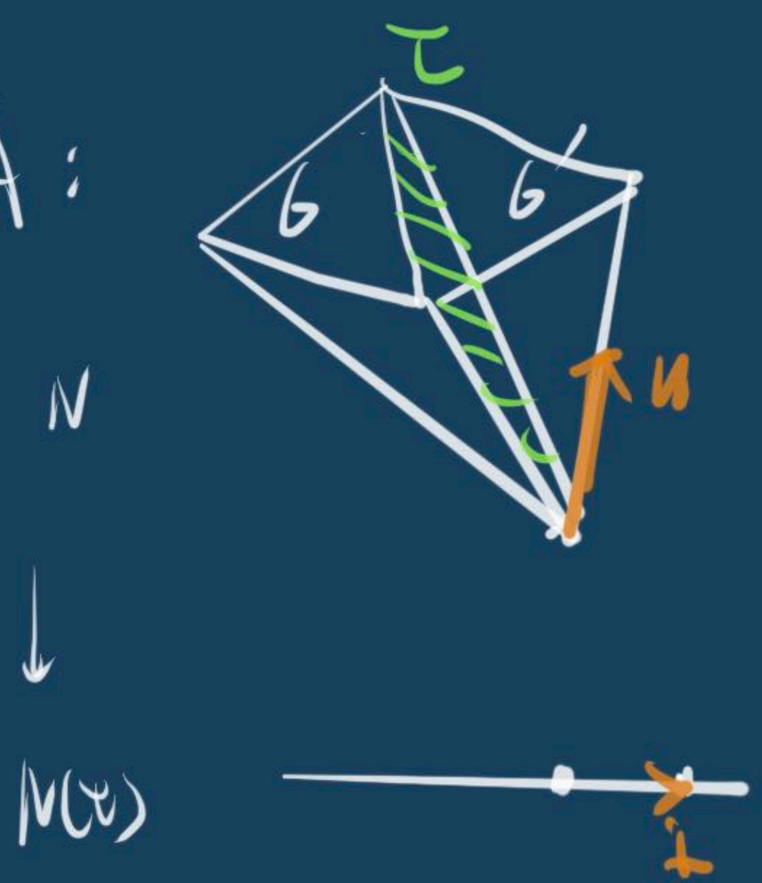
Toric version: if $X = X_\Sigma$.

Thm. 1). D is nef $\iff D \cdot V(\sigma_1) \geq 0 \ \forall \ \sigma_1 \in \Sigma(n-1)$.

2). D is ample $\iff D \cdot V(\sigma_1) > 0$, " "

Q: What is $D \cdot V(\sigma_1)$?

A:



$\sigma \cap \sigma' = \tau$

$N_\tau :=$ lattice gen by τ.

$N(\tau) = N / N_\tau$.

Cartier data of D on σ, σ' are $m_\sigma, m_{\sigma'}$

$\Rightarrow \quad D \cdot V(\tau) = \langle m_\sigma - m_{\sigma'}, u \rangle$.