Weighted proj spaces revisited:

alg: $k^*_\lambda \curvearrowright \mathbb{A}^{n+1}\setminus\{0\}$ via

$$\lambda\cdot(x_0,\dots,x_n) = (\lambda^{d_0}x_0,\dots,\lambda^{d_n}x_n)$$

$d_0,\dots,d_n \in \mathbb{N}$.

wt: $\vec{d}=(d_0,\dots,d_n)$

$$\frac{\mathbb{A}^{n+1}\setminus\{0\}}{k^*_\lambda} =: \mathbb{P}^n(d_0,\dots,d_n)=\mathbb{P}^n(\vec d)$$

observation: $\vec d$ and $k\vec d$

$$\mathbb{P}(\vec d)\cong\mathbb{P}(k\vec d)$$

WLOG, may assume $\gcd(d_0,\dots,d_n)=1$. (*)

Coord. ring of $\mathbb{P}(\vec d)$ is

$$k[x_0,\dots,x_n]$$

$$\deg x_0^{a_0}\cdots x_n^{a_n} = a_0d_0+\dots+a_nd_n$$

Toric construction:

Assume (*): $N := \mathbb{Z}^{n+1}/\mathbb{Z}(d_0,\dots,d_n) \leftarrow \mathbb{Z}^{n+1}$

$u_i \mapsfrom e_i$

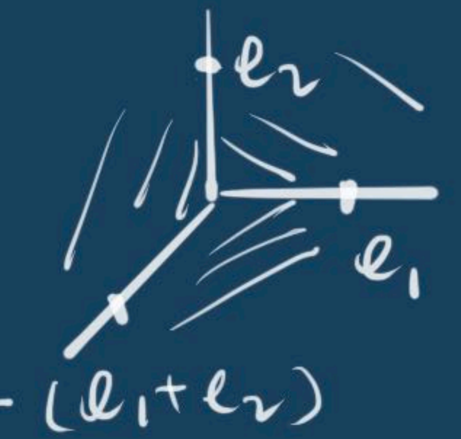
$$\Rightarrow \sum_{i=0}^{n} d_i u_i = 0$$

$\rightsquigarrow \Sigma$: fan gen by proper subsets of $\{u_0,\dots,u_n\}$

$\Rightarrow X_\Sigma = \mathbb{P}^n(d_0,\dots,d_n)$

e.g. $\underbrace{\Sigma}_{?}$ for $\mathbb{P}^n = X_\Sigma$

known: $n=2$:



in general: $\Sigma =$ gen. by

$e_1, e_2, \dots, e_n, -(e_1+\dots+e_n)$

Rmk: this can be see immediately by using Cox's construction of toric vars by quotient.

$X = X_\Sigma$, $\Sigma(1)$: dim 1 cones.

$$\overset{\text{Cox}}{\Rightarrow} X \cong \mathbb{A}^{|\Sigma(1)|} - Z \Big/ \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), k^*)$$

Rmk: in general, $\mathbb{P}(\vec d)$ is sing.

Birational geo. of toric var's (I)
(not involving MMP)

last time we saw:

$\exists X_\Sigma \begin{cases} \text{complete (sm) toric var's} \\ \text{not proj.} \end{cases}$

In general:

Chow lemma: $\exists\ \tilde{Y} \xrightarrow[\text{proper}]{\text{bir}} Y$ complete var.

w/ $\tilde{Y}$ proj

Toric Chow lemma: X_Σ toric, $\exists\ X_{\tilde\Sigma} \xrightarrow{f} X_\Sigma$ s.t.

$\begin{cases} \tilde\Sigma \text{ is a subdivision of } \Sigma \\ X_{\tilde\Sigma} \text{ is quasi-proj} \\ f: \text{bir. proper.} \end{cases}$

in particular, X_Σ is complete
$\Downarrow$
$X_{\tilde\Sigma}$ is proj.

Q: What is "birational" in toric picture?
"proper" ($\varphi^{-1}(|\Sigma'|) = |\Sigma|$)

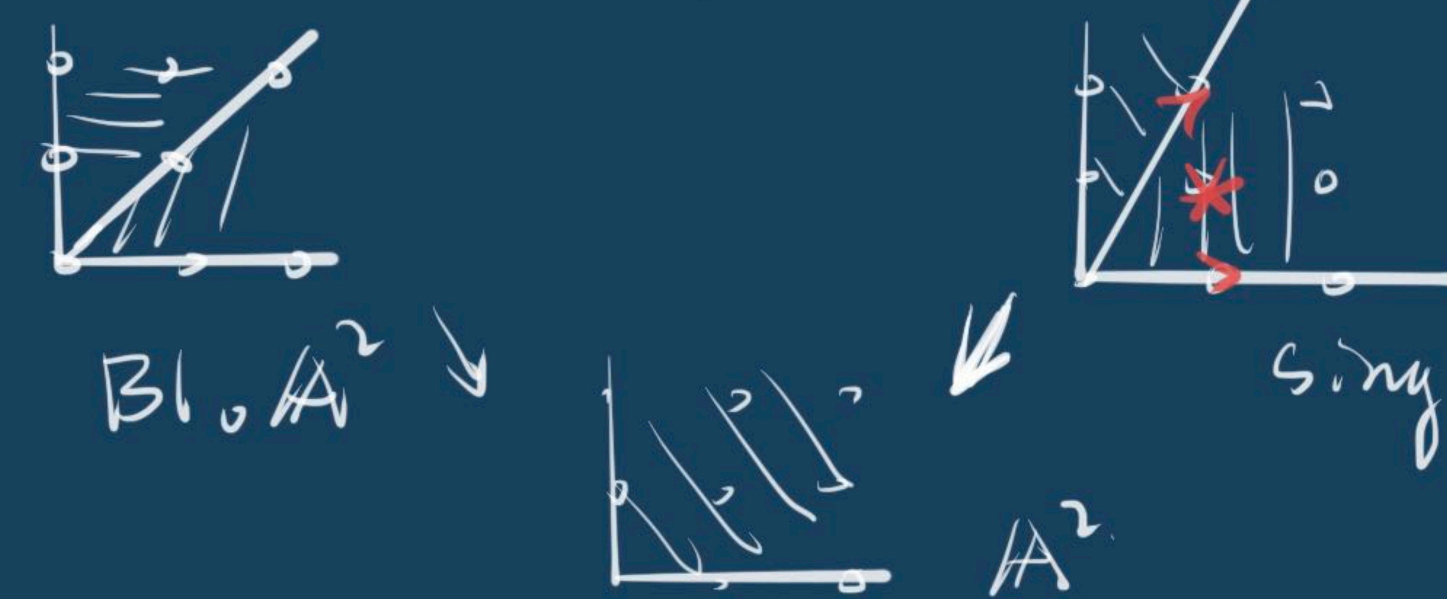
We need:
$$\begin{array}{ccc} T_N & \subseteq & X_{\tilde\Sigma} \\ \parallel & & \downarrow \\ T_N & \subseteq & X_\Sigma \end{array}$$

by orb-cone corresp:

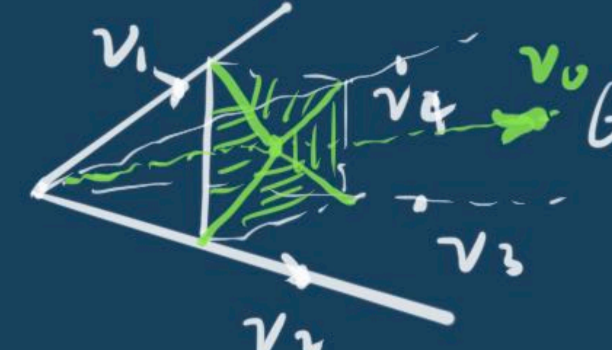
- N needs to be fixed.
- $\tilde\Sigma$ should be finer than Σ.

How to subdivide?

e.g. $N_{\mathbb{R}}$



Blow up @ a T-invariant pt



T-inv. pt P
$\updownarrow$ orb-cone
σ° $\dim \sigma^\circ = \dim T$

$\overline{\sigma^\circ} = \sigma$, $\sigma(1) = \{\rho_1, \cdots, \rho_k\}$

v_i : primitive vector on ρ_i

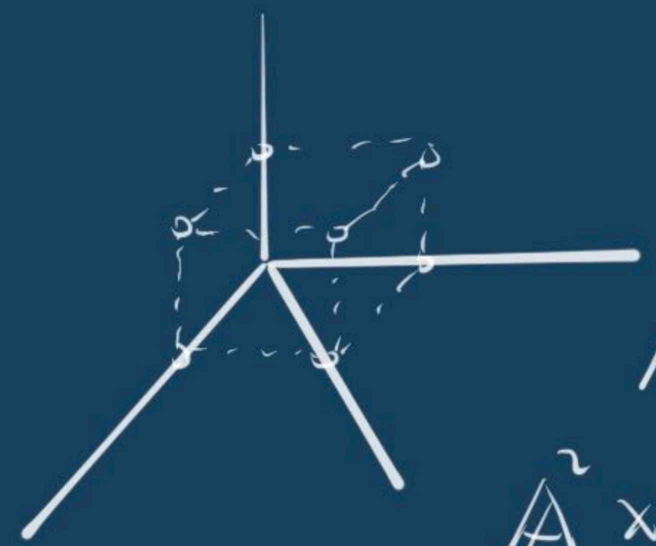
let $v_0 = v_1 + \cdots + v_k$.

subdiv. σ by adding $v_0 \rightsquigarrow \mathrm{Star}(\sigma)$: new fan

$Bl_p\, \mathcal{U}_\sigma \longrightarrow \mathcal{U}_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$
$\parallel$
$X_{\mathrm{Star}(\sigma)}$

Rmk: if $v_0 = d_1 v_1 + \cdots + d_k v_k$.
$\rightsquigarrow$ wted blowing up

Blow up along higher dim toric subvar's:

e.g. 

$\mathbb{A}^3$

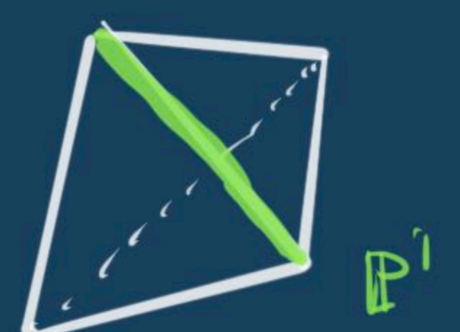
$$\mathbb{A}^2 \leftarrow Bl_0 \mathbb{A}^2$$

$$\mathbb{A}^2 \times \mathbb{A}^1 \leftarrow (Bl_0 \mathbb{A}^2) \times \mathbb{A}^1$$

$$\quad \| \qquad\qquad\qquad \|$$

$$\mathbb{A}^3 \leftarrow Bl_{\mathbb{A}^1} \mathbb{A}^3$$

or on the dual side say looking at $\mathbb{P}^3$



$\mathbb{P}^3$

$\leftarrow$

$Bl_{\mathbb{P}^1} \mathbb{P}^3$

in general: $\tau \in \Sigma$ not full dim $\rightsquigarrow$ Star(τ)

$$\downarrow$$

$$\Sigma' \to \Sigma$$

$\tau \leftrightarrow O_\tau \subseteq V(\tau) = \overline{O_\tau}$

orb. cone

$$X_{\Sigma'} \longrightarrow X_\Sigma$$

$$\|$$

$$Bl_{V(\tau)} X_\Sigma$$

Back to Chow lemma.

Pf (complete $\Leftarrow$ proj)

$X = X_\Sigma$ complete $|\Sigma| = N_{\mathbb{R}}$, want: $\begin{cases} \text{subdivide } \Sigma \rightsquigarrow \Sigma' \text{ finer} \\ \text{strong convex pw. linear} \\ \text{function on } \Sigma'. \end{cases}$

$\sigma \in \Sigma(n-1)$, choose $m_\sigma \in M$ s.t.

$\sigma \subseteq H_{m_\sigma} = \{ n \in N_{\mathbb{R}} \mid \langle n, m_\sigma \rangle = 0 \}$

for all σ in $\Sigma(n-1)$

$\rightsquigarrow \underbrace{\{H_{m_\sigma}\}_{\sigma \in \Sigma(n-1)} \text{ hyperplane arr.}}$

form a finer fan than Σ.

$$\|$$

$$\Sigma'$$

$$X_{\Sigma'} \xrightarrow{\text{proper}} X_\Sigma$$

the function we expect:

$$\psi(u) = -\sum_{\sigma \in \Sigma(n-1)} |\langle u, m_\sigma \rangle| \text{ is}$$

$\Downarrow$ actually strongly convex.

ample l.b. $\Rightarrow X_{\Sigma'}$ proj.

Thm (Moishezon) $/\mathbb{C}$

X : sm bimero. to proj mfd

$\Rightarrow \exists\ \tilde{X} \xrightarrow{\pi} X$ w/ $\tilde{X}$: sm. proj

π : a seq. of blowing up along sm. centers.

Toric version:

(T-Mo) X : sm. complete toric

$\Rightarrow \exists\ \tilde{X} \xrightarrow{\pi} X$ w/ $\tilde{X}$: sm proj toric

π : blowing ups along sm T-inv centers of codim 2.

Thm (De Concini - Procesi)

X, X' sm toric, $f : X \overset{\text{bir}}{\dashrightarrow} X'$

$\Rightarrow \exists$ lift $\tilde{X}$ sm

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi"] & \\ X \arrow[r, dashed, "f"'] & X' \end{tikzcd}$$

s.t. π: bl. ups along codim 2 T-inv. centers

$f\circ\pi$: regular.

Pf: DC-P $\Rightarrow$ T-Moi $\quad \dim X = n$

$\exists\ f : \mathbb{P}^n \dashrightarrow X$ because T_N

DC-P $\Rightarrow$

$$\begin{tikzcd} \bar{X} \arrow[d, "\bar{\pi}"'] \arrow[dr, "f\circ\bar{\pi} =: \bar{f}"] & \\ \mathbb{P}^n \arrow[r, dashed, "f"'] & X \end{tikzcd}$$

w/ $\bar{X}$: sm. proj

$\bar{\pi}$: bl. ups along codim 2 T-inv. centers.

DC-P $\Rightarrow$

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"'] \arrow[dr, "h"] & \\ X \arrow[r, dashed, "\bar{f}^{-1}"'] & \bar{X} \end{tikzcd}$$

h: bir. reg.

π is what we want.

Toric surf. sing's & resolutions.

$\mathbb{R}^2 \simeq N_{\mathbb{R}} \supset \sigma$ cone $\Longrightarrow$ $\exists\, e_1, e_2$: basis of N

Claim s.t. $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$

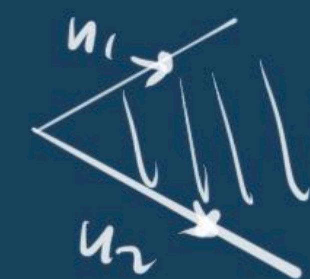
w/ $0 \le k < d$, $(d,k) = 1$.

(coord. changing)

Why?

$\sigma = \mathrm{Cone}(u_1, u_2)$

primitive



Step 1:

basis	$e_1 = e_1'$	$e_2 = u_1$

can find since u_1 is primitive.

Step 2: fact: for any $\begin{cases} l \in \mathbb{Z} \\ d \in \mathbb{Z}_+ \end{cases}$, $l = sd - k$, $0 \le k < d$ uniquely.

(*) $u_2 = de_1' + le_2$ $\overset{\text{fact}}{\Longrightarrow}$ $\exists!\, s,k$ s.t. $l = sd - k$, $0 \le k < d$

$d \neq 0$, $d > 0$ (replace e_1 by $-e_1$ if necessary).

basis	$e_1 = e_1' + se_2$	$e_2 = u_1$

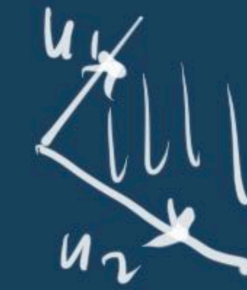
by (*) $u_2 = d(e_1 - se_2) + le_2$

$= de_1 + (l - sd)e_2$

$= de_1 - ke_2$

$\gcd(d,k) = 1 \Leftarrow u_2$ primitive.

For a cone σ:



$\rightsquigarrow$ lattice N' gen by u_1, u_2

$\cap$

$N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$

by the claim:

$N' = \mathbb{Z}e_2 \oplus \mathbb{Z}(de_1 - ke_2)$

$= d\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$

$0 \to N' \to N \to N/N' \to 0$

$N/N' = G \simeq \mathbb{Z}/d\mathbb{Z}$

$U_\sigma = U_{\sigma,N} = \mathrm{Spec}\, k[\sigma^\vee \cap M]$

$N' \hookrightarrow N \rightsquigarrow \underbrace{M' \hookleftarrow M}$

$\mathbb{C}[\sigma^\vee \cap M'] \hookleftarrow \mathbb{C}[\sigma^\vee \cap M]$

$\mathrm{Spec}\,\mathbb{C}[\sigma^\vee \cap M'] \longrightarrow \mathrm{Spec}\,\mathbb{C}[\sigma^\vee \cap M]$

$\mathbb{C}^2 \twoheadrightarrow U_\sigma$

it's: $\mathbb{C}^2/G \simeq U_\sigma$

Goal: · understand/classify $\mathbb{C}^2/G\ (\simeq U_\sigma)$

· how to resolve the sing's?

$\mu_d := \{\zeta \mid \zeta^d = 1\} \simeq \mathbb{Z}/d\mathbb{Z}\ (\simeq G = N/N')$, $\zeta \in \mathbb{C}$

$G \curvearrowright \mathbb{C}^2 \iff \mu_d \curvearrowright \mathbb{C}^2$ via

$\zeta \cdot (x, y) = (\zeta x, \zeta^k y)$ ⟵ why? (Claim)

$\rightsquigarrow \mathbb{C}^2 / \mu_d$ called type $\frac{(1,k)}{d}$.

e.g. $N' \hookrightarrow N$, $N' = \mathbb{Z}^2$ $\qquad$ $M' \hookleftarrow M$, $M' = \mathbb{Z}^2$

N

$\frac{(1,1)}{2}$

M

Q: Fix $N' = \mathbb{Z}^2$, what is N?

A: $N = N' + \mathbb{Z} \cdot \frac{(1,k)}{d}$.

$(x,y) \mapsto (\zeta x, \zeta^k y)$ invariant monomials?

$x^a y^b = \zeta^{a+kb} x^a y^b$

$\Updownarrow$

$d \mid a + kb$

$\Updownarrow$

$\frac{(1,k)}{d} \cdot (a, b) \in \mathbb{Z}^2 = N'$

Observation: $\Leftarrow$ ✓

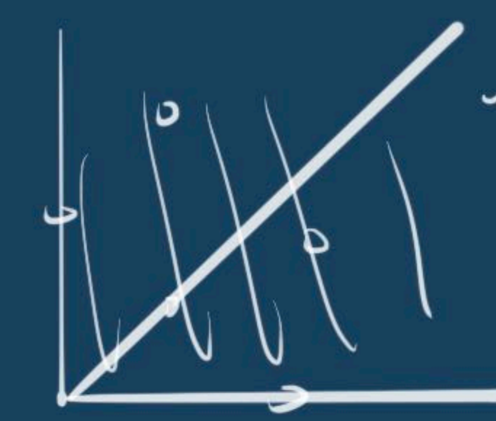
$k_1 k_2 \equiv 1 \pmod d \Rightarrow \frac{(1,k_1)}{d} = \frac{(1,k_2)}{d}$

$(x,y) \mapsto (\zeta x, \zeta^{k_1} y) = (\eta^{k_2} x, \eta y)$

let $\eta = \zeta^{k_1} \Rightarrow \eta^{k_2} = \zeta^{k_1 k_2} = \zeta$

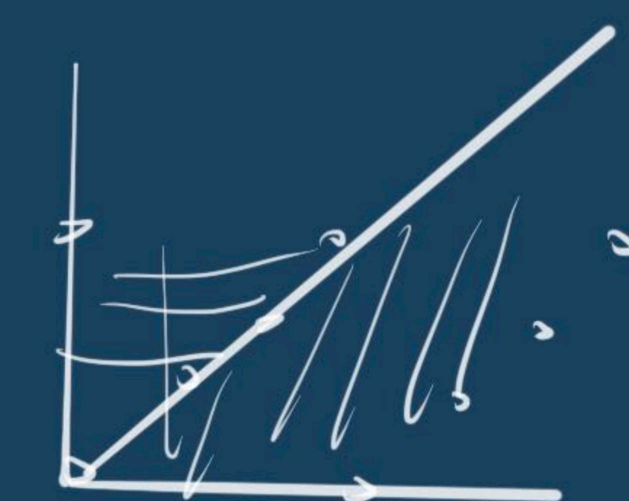
Thm. $\mathcal{U}_{\sigma_1} = \mathcal{U}_{\sigma_2} \iff \begin{cases} d_1 = d_2 \\ \text{either } k_1 = k_2 \\ \quad \text{or } k_1 k_2 \equiv 1 \pmod d \end{cases}$

e.g.



resolution

$\frac{(1,1)}{2}$



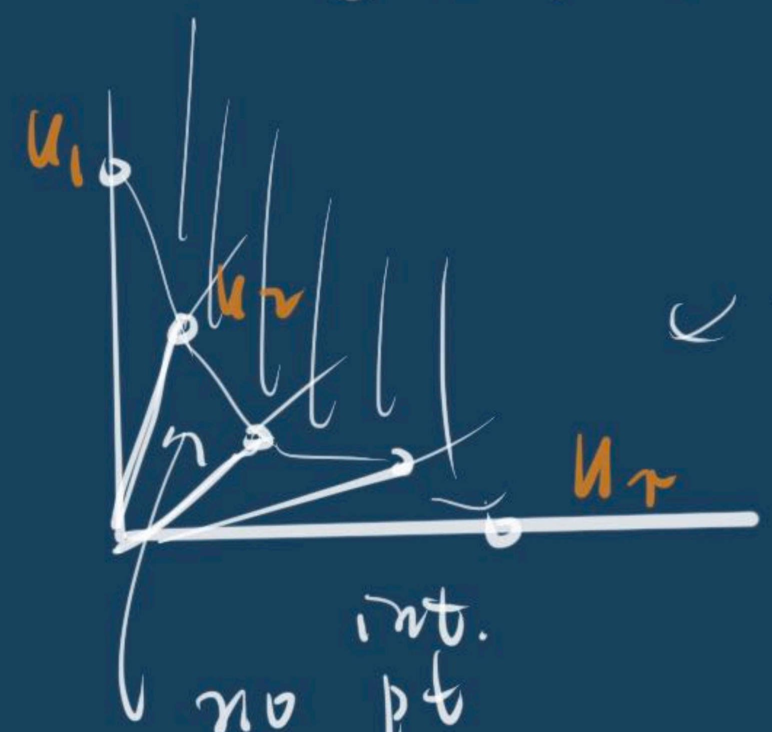
$\frac{(1,1)}{3}$

$\frac{(1,2)}{3}$

Thm. $\mathbb{Z}^2 \subseteq N$, σ: 1-quadrant
$\overset{\shortparallel}{N'}$

The min. resolution of U_σ corresp.

$\mathrm{Conv}(N \setminus \{0\}) \subseteq$ 1-quadrant.



← this is sm.

to pf: need $\mathbb{Z}(u_i, u_{i+1}) = N$

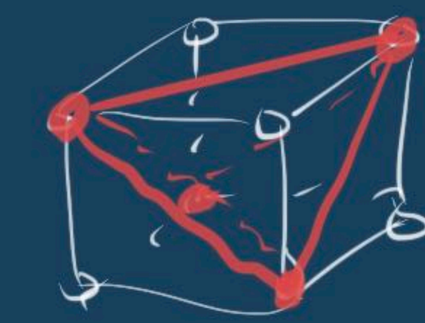
Pick's lemma: Q polytope.

$A(Q)$ = area of $Q \cdot 2! = $ (# vertices of Q)
normalized area
+ (# $\mathbb{Z}$-pts on ∂Q)
+ 2(# $\mathbb{Z}$-pt interior)
-2.

Cor. Q is a triangle only $\mathbb{Z}$ pts on Q are vertices $\iff A(Q) = 1 \Rightarrow$ gen.

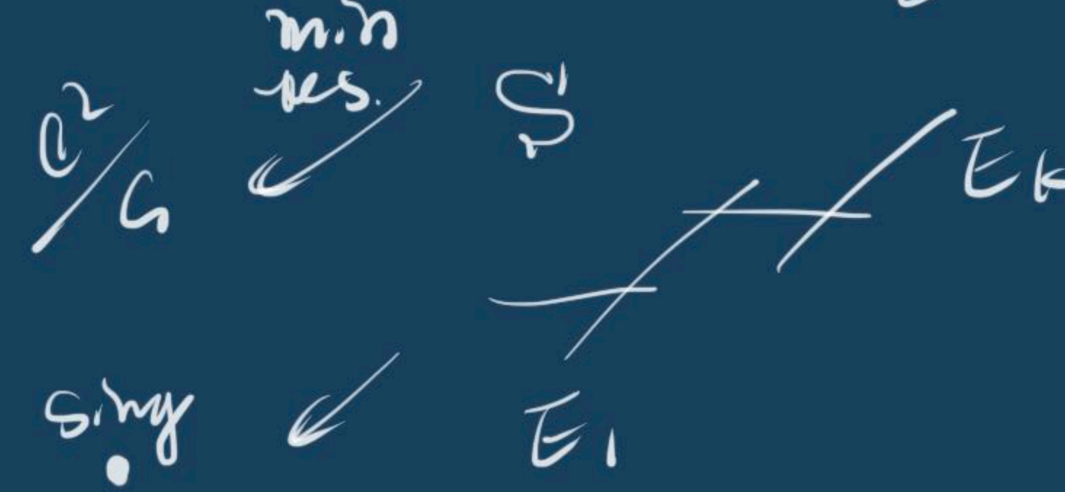
the lattice only true in dim 2.

in dim 3:



vol = 1.

What else can we say for the res.?



always type A dual complex:

$\mathbb{P}^1 \cong E_i$, $E_i^2 \leq -2$ (min)

Q: what is this? $-E_i^2 = m_i$

A: Hirzebruch-Jung cont. frac's

$$\frac{n}{a} = m_1 - \frac{1}{m_2 - \dots} \qquad m_1 = \left\lceil \frac{n}{a} \right\rceil$$

e.g. $\frac{(1,4)}{7}$: $\frac{7}{4} = 2 - \frac{1}{4}$

E_1 2
E_2 4

$\frac{(1,6)}{7}$: $\frac{7}{6} = 2 - \frac{1}{\frac{6}{5}}$

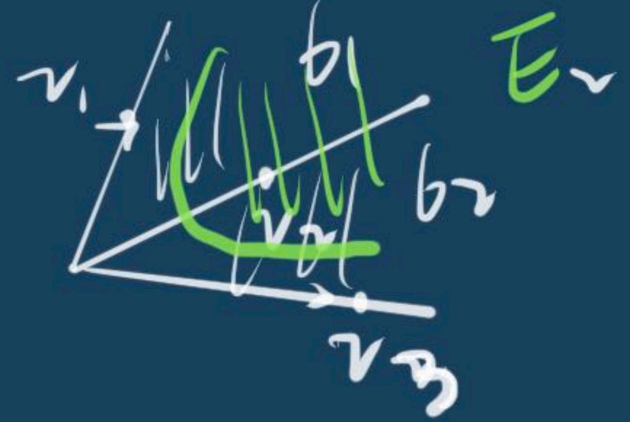
$= 2 - \frac{1}{2 - \frac{4}{5}}$

$\vdots$

$= 2 - \frac{1}{2 - \dots}$

2
2
2
2
2
2

Reason (to be updated).

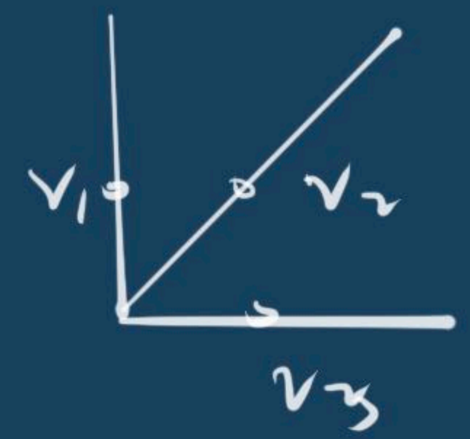


$a v_2 = v_1 + v_3$

$\Downarrow$

$E_2^2 = -a$.

e.g.



$1 \cdot v_2 = e_1 + e_2$

$= v_3 + v_1$

$E^2 = -1$.

Rmk: · in dim 3 we lose all these.

· instead of $G = \operatorname{coker}(N' \to N)$

generalization: finite subgps of $SL_2(\mathbb{C})$
$\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad\qquad H$.

$H \curvearrowright \mathbb{C}^2 \rightsquigarrow \mathbb{C}^2/H$. $\longleftrightarrow$ dual complex must be a ADE type Dynkin diagram.