Previously: normal var's.

- affine toric var's $\leftrightarrow$ rational f.g. cone $\subseteq N_{\mathbb{R}}$
- proj toric var's $\leftrightarrow$ lattice polytope in $M_{\mathbb{R}}$ or fan Σ in $N_{\mathbb{R}}$ from $Q \subseteq M_{\mathbb{R}}$ i.e. $\Sigma = \Sigma_Q$ ($\uparrow$ normal fan.)
  = toric var + ample l.b.

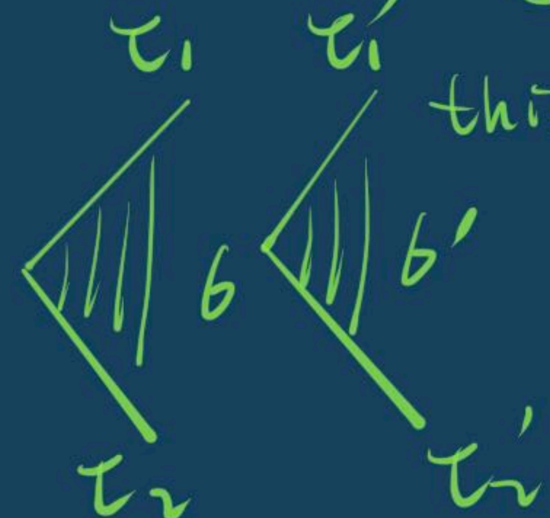
In general: arbitrary toric var's — not necessarily affine / proj.

Construction:

glue cones $\rightsquigarrow$ a fan $\rightsquigarrow X_\Sigma$ is irr. reduced normal toric.

this is a require-

non e.g.



glueing: $\tau_1 \rightsquigarrow \tau_1'$, $\tau_2 \rightsquigarrow \tau_2'$

this is not a fan in $N_{\mathbb{R}}$.

Thm. Any irr. toric var X must be X_Σ for some fan Σ.

Notation: $|\Sigma| = \operatorname{supp}\Sigma = \bigcup_{\sigma\in\Sigma} \sigma \subseteq N_{\mathbb{R}}$

Thm (orb-cone corresp.)

Fix $\Sigma \subseteq N_{\mathbb{R}} \rightsquigarrow X_\Sigma$.

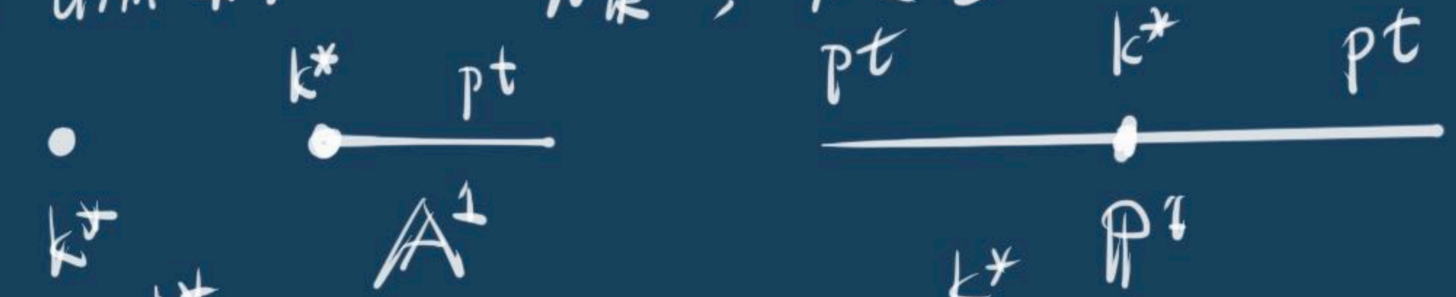
$$|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma = \coprod_{\sigma\in\Sigma} \sigma^\circ$$ (picture is inverted)

$\updownarrow$ 1:1

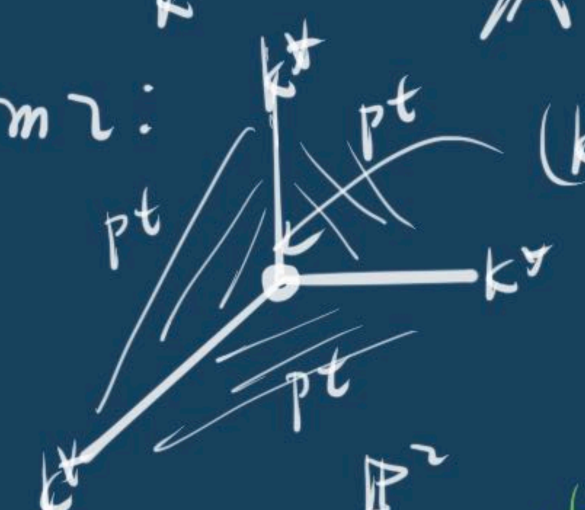
$$X_\Sigma = \coprod T_N\text{-orb.}$$

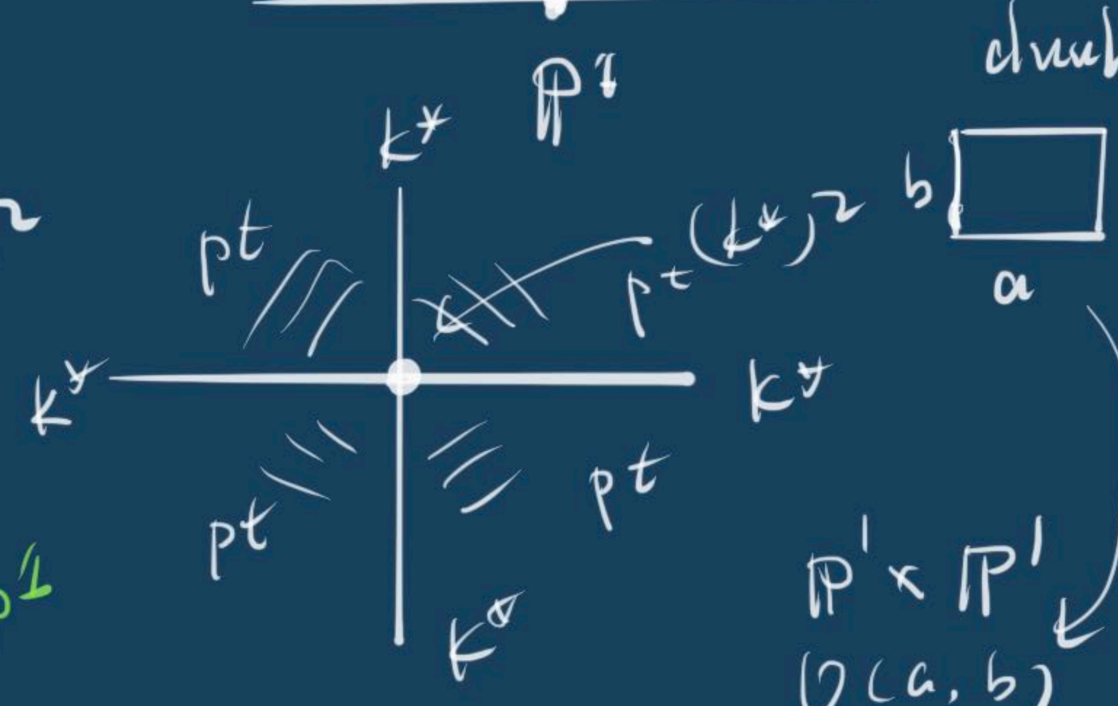
$\dim \sigma^\circ$ ($\dim\sigma$) $= n - \dim T_N\text{-orb.}$, $n = \operatorname{rank} N$.

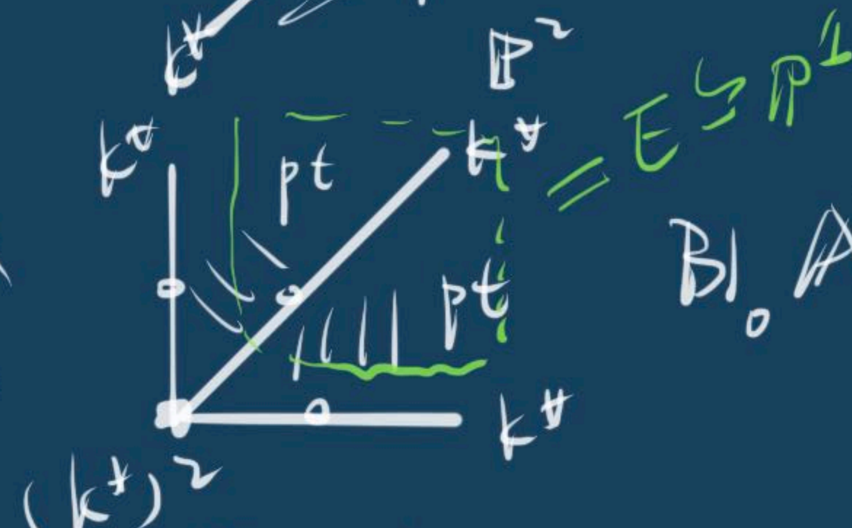
e.g. dim 1: $N_{\mathbb{R}}$, $N \simeq \mathbb{Z}$



dim 2:







Q: Does the glueing break separatedness?

A: No.

Separatedness

Def: A scheme X is separated if

$\Delta : X \to X \times_{\mathrm{Spec}\, k} X$ is a closed immer (→ Spec k)

(more general: $X \xrightarrow{} Y$ sep. if $X \xrightarrow{\Delta} X \times_Y X$ closed imm.)

Want: limit pt unique if it exists.

pt set top: # limit pt $\le 1 \Leftrightarrow$ Hausdorff + 1-count.

Zariski top not

~~Standard~~ non-eg: "double headed snake"

glue two $\mathbb{A}^1$'s along k^*:

benefit: sep $\Rightarrow$ (affine $\cap$ affine) is affine

$\Leftarrow$ + $\Gamma(U, \mathcal{O}_U) \otimes \Gamma(V, \mathcal{O}_V)$ (s,t)

$\twoheadrightarrow \Gamma(U \cap V, \mathcal{O}_{U\cap V})$ $\quad s|_{U\cap V}\ t|_{U\cap V}$

$\mathbb{A}^2_{x,y}$ glue away from 0

$\mathbb{A}^2_{z,w}$

not sep.

Valuative criterion:

alg: $\forall$ DVR R w/ frac$(R) = K$

$$\begin{array}{ccc} \mathrm{Spec}\, K & \longrightarrow & X \\ \downarrow & \overset{\# \le 1}{\nearrow} & \downarrow \\ \mathrm{Spec}\, R & \longrightarrow & Y \end{array}$$

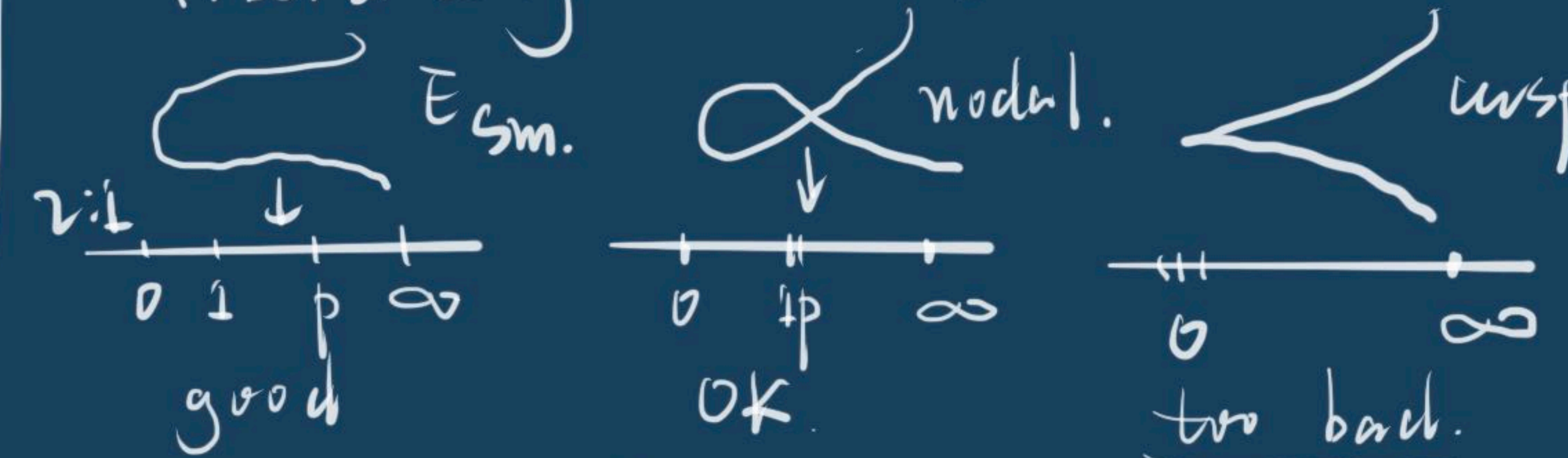
geometrically:

From moduli pt of view:

$\overline{M}_g \supseteq M_g :=$ moduli of sm. genus $g \ge 2$ alg curves

on the boundary: $\partial \overline{M}_g = \overline{M}_g \setminus M_g$

fibers (degenerations): nodal curves



$C_t : y^2 = x^3 + t^6$ $\qquad t \ne 0 : C_t \cong C_t'$

$C_t' : y^2 = x^3 + 1$ $\qquad t = 0 : C_0 \not\cong C_0'$

$\overline{M}_g^{\text{cusp}}$ is not sep. t

Rmk: Serre's GAGA

X sep $\Leftrightarrow$ X^{an} Hausdorff.

Prop. X toric $\Rightarrow$ separated

gluing of cones to a fan $\underset{\text{only}}{\rightsquigarrow}$ sep. toric var's

Rmk: in toric stack case, this is not always true.

Recall: $U_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$

Pf: need:

$\Delta: X \to X \times X$ to be a closed emb.

locally: $\sigma_1, \sigma_2 \in \Sigma$, $\tau = \sigma_1 \cap \sigma_2$

need: $U_\tau \longrightarrow U_{\sigma_1} \times U_{\sigma_2}$ closed emb.

$\Updownarrow$

alg: $k[\tau^\vee \cap M] \xleftarrow[\Delta^*]{} k[\sigma_1^\vee \cap M] \otimes k[\sigma_2^\vee \cap M]$

$\chi^{m_1+m_2} \xleftarrow{} \chi^{m_1} \otimes \chi^{m_2}$

need Δ^* surj. $\Leftarrow$ $S_\tau = S_{\sigma_1} + S_{\sigma_2}$ $(*)$

$(*)$: because

- $\sigma_1^\vee + \sigma_2^\vee = (\sigma_1 \cap \sigma_2)^\vee = \tau^\vee \Rightarrow$ " $\supseteq$ "
- $p \in S_\tau$ previously $S_\tau = S_\sigma + \mathbb{Z}(-m)$

$M \ni m \rightsquigarrow H_m = \{ n \in N_{\mathbb{R}} \mid \langle n, m \rangle = 0 \}$

s.t. $\tau = H_m \cap \sigma$.

choose $m \in M$ s.t. $\tau = \sigma_1 \cap H_m = \sigma_2 \cap H_m$

e.g. $m \in \sigma_1^\vee \cap (-\sigma_2^\vee) \cap M$

$$\underset{\tau}{p} = \underset{S_{\sigma_1}}{q} + \underset{S_{\sigma_2}}{l(-m)} \quad \text{since } (-m) \in \sigma_2^\vee$$

$\Rightarrow p \in S_{\sigma_1} + S_{\sigma_2}$.

Toric morphisms:

$$\begin{array}{ll} X_{\Sigma'} & : T_{N'}\text{-toric var.} \\ \downarrow & \\ X_\Sigma & : T_N\text{-toric var.} \end{array}$$

can be arbitrary

Want: not all morphisms

- $T_{N'}$-orb $\longrightarrow$ T_N-orb.

On geometry:

Def: $\varphi: X_{\Sigma'} \to X_\Sigma$ is toric if

$$\begin{array}{ccc} X_{\Sigma'} & \to & X_\Sigma \\ \cup\!\! \mid & & \cup\!\! \mid \\ T_{N'} & & T_N \end{array}$$

$\varphi(T_{N'}) \subseteq T_N$ and $\varphi|_{T_{N'}}: T_{N'} \to T_N$ is a gp hom.

On combinatorics:

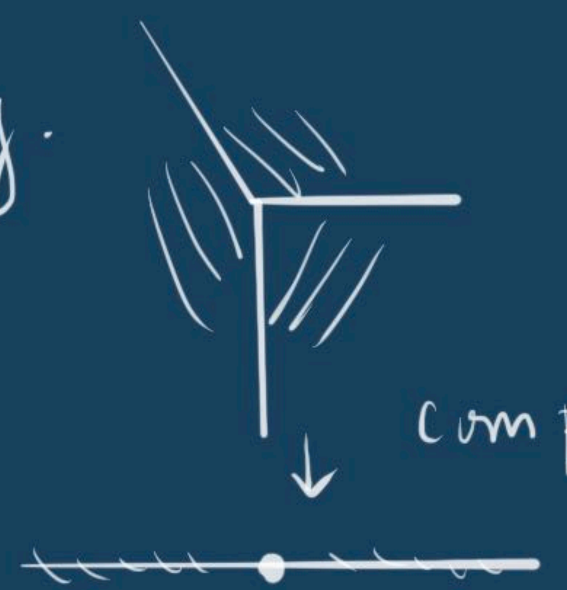
Def: $\Phi: N' \longrightarrow N$ $\mathbb{Z}$-linear.

$\rightsquigarrow \Phi_{\mathbb{R}}: N'_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}$

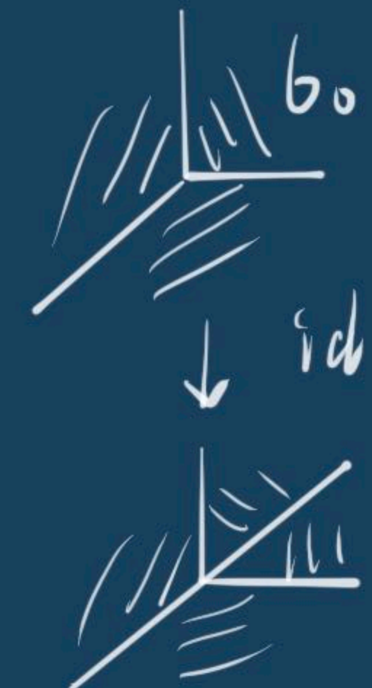
$$\underset{\Sigma'}{} \quad \underset{\Sigma}{}$$

Φ is compatible w/ Σ', Σ if $\forall \sigma' \in \Sigma'$, $\exists \sigma \in \Sigma$ s.t. $\Phi_{\mathbb{R}}(\sigma') \subseteq \sigma$.

eg.



noneg.



$\Phi(\sigma_0)$ is not in any cone

(no morphism from $\mathbb{P}^2$ to $Bl_1\mathbb{P}^2$)

Thm: $\varphi: X_{\Sigma'} \to X_\Sigma$ toric morphism

$\Updownarrow$

$\Phi: N' \to N$ compatible w/ Σ', Σ.

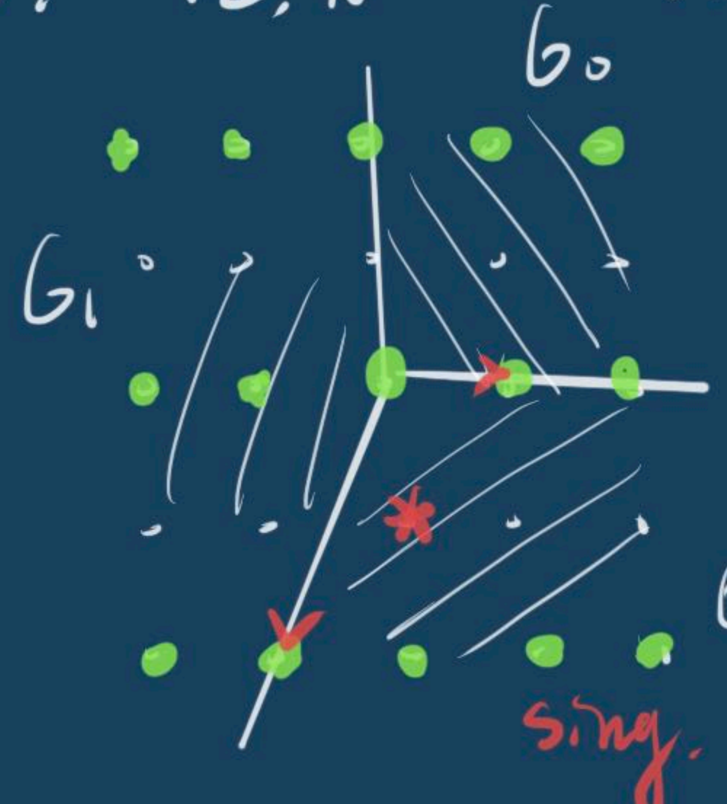
Special interesting case:

$N' \hookrightarrow N$ w/ finite index (the same rank)

$\Sigma \subseteq N'_{\mathbb{R}} = N_{\mathbb{R}}$

$\varphi: X_{\Sigma,N'} \longrightarrow X_{\Sigma,N}$ toric morphism.

e.g.



$N \longleftrightarrow N' = \{(a,b) \in N \mid b \text{ even}\}$

view Σ in $N'_{\mathbb{R}}$

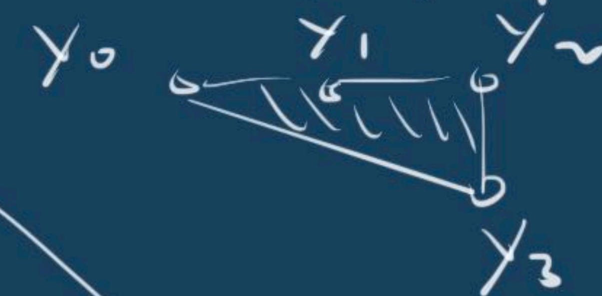
$X_{\Sigma,N'} = \mathbb{P}^2$

view Σ in $N_{\mathbb{R}}$

$X_{\Sigma,N}$ is sing.

What is this?

normal polytope (up to a choice)



$\operatorname{Proj} \dfrac{k[y_0, \dots, y_3]}{(y_0y_2 - y_1^2)}$

$\cap$

$\mathbb{P}^3_{y_0,\dots,y_3}$

$\mathbb{P}(1,1,2)$

$\|$

$[(\mathbb{A}^3 \setminus \{0\})/k^*]_2$: $\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$

$[(\mathbb{A}^3 \setminus \{0\})/k^*]_1 = \mathbb{P}^2$ $\quad G = N/N' = \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z} \quad \big\downarrow 2:1$

$[\mathbb{A}^3 \setminus \{0\}/k^*]_2 = \mathbb{P}(1,1,2)$

Prop. $N' \hookrightarrow N$, Σ fan in $N'_{\mathbb{R}} = N_{\mathbb{R}}$

$\varphi: X_{\Sigma,N'} \longrightarrow X_{\Sigma,N}$ is a quotient map. i.e. $X_{\Sigma,N} = X_{\Sigma,N'}/G$

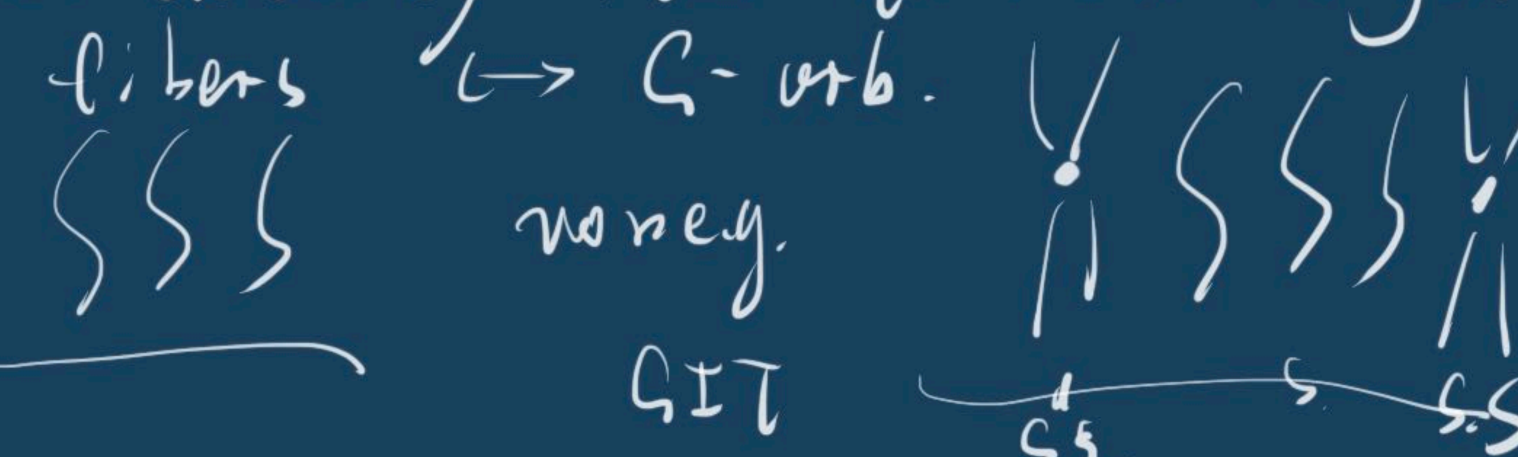
$G = N/N'$.

Pf: need in the dual lattice:

$k[\sigma^\vee \cap M']^G = k[\sigma^\vee \cap M]$

Rmk: actually, the quotient is geo.

fibers $\longleftrightarrow$ G-orb.

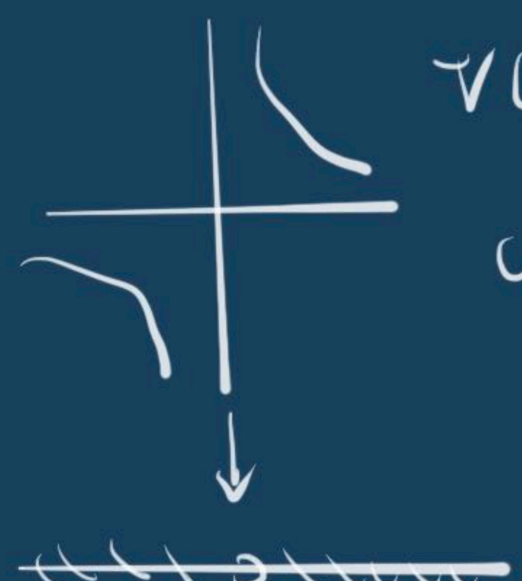


Properness:

Def: $X \xrightarrow{f} Y$ is proper if f is $\begin{cases} \text{sep.} \\ \text{finite type} \\ \text{universal closed.} \end{cases}$

e.g. $\mathbb{P}^1 \times \mathbb{P}^1$

non. $\mathbb{A}^1_k \to \mathrm{pt}$ is sep. finite.



$V(xy-1)$ $=$ closed $\downarrow$ non closed

$$\begin{array}{ccc} \mathbb{A}^2_k & \to & \mathbb{A}^1_k \\ \downarrow & & \downarrow \\ \mathbb{A}^1_k & \to & \mathrm{pt} \end{array}$$

Def: $X \to \mathrm{Spec}\, k$ proper $\Rightarrow$ call X complete.

Valuative criterion:

alg:
$$\begin{array}{ccc} \mathrm{Spec}\, K & \to & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}\, R & \to & Y \end{array}$$
exact one

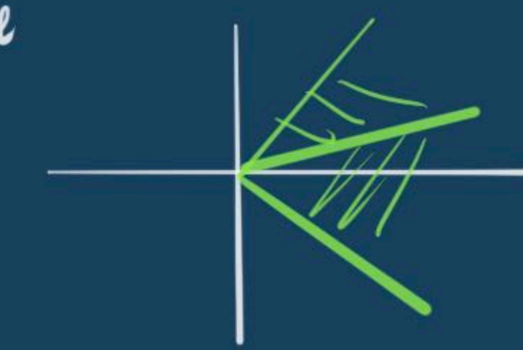
geo:

Thm. $\varphi: \Sigma' \to \Sigma \rightsquigarrow \varphi_*: X_{\Sigma'} \to X_\Sigma$

$\varphi: N'_{\mathbb{R}} \to N_{\mathbb{R}}$

φ_* is proper $\Leftrightarrow$ $\varphi^{-1}(|\Sigma|) = |\Sigma'|$.

Cor. X_Σ is complete $\Leftrightarrow$ $|\Sigma| = N_{\mathbb{R}}$.



non eg.

$\mathbb{A}^1$

$\downarrow \varphi$ not proper.

Intuitive pf of the cor.

$\varphi^{-1}(\bullet\!\!-\!\!-) =$

$\times$

Rmk: X complete $\overset{\text{GAGA}}{\Leftrightarrow}$ X^{an} compact

complete want limit $\exists$

$\subset M_{\mathbb{R}}$

by orb-cone.

pt

pt

k^* taking limit

$(k^*)^2$

pt not in the var.

k^*

Recall: complete $= (|\Sigma| = N_{\mathbb{R}})$

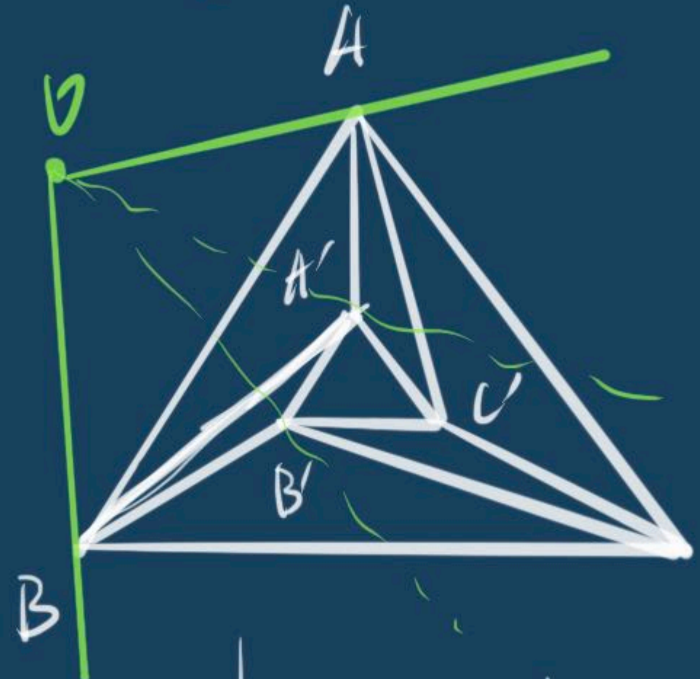
proj $= \exists$ ample l.b.

$\Updownarrow$

$\exists$ str. convex function over Σ.

expect: Σ does not admit str. convex fnc's.

e.g.



$AB /\!/ A'B'$
$AC /\!/ A'C'$
$BC /\!/ B'C'$
$\rightsquigarrow$ complete
not proj
sing.

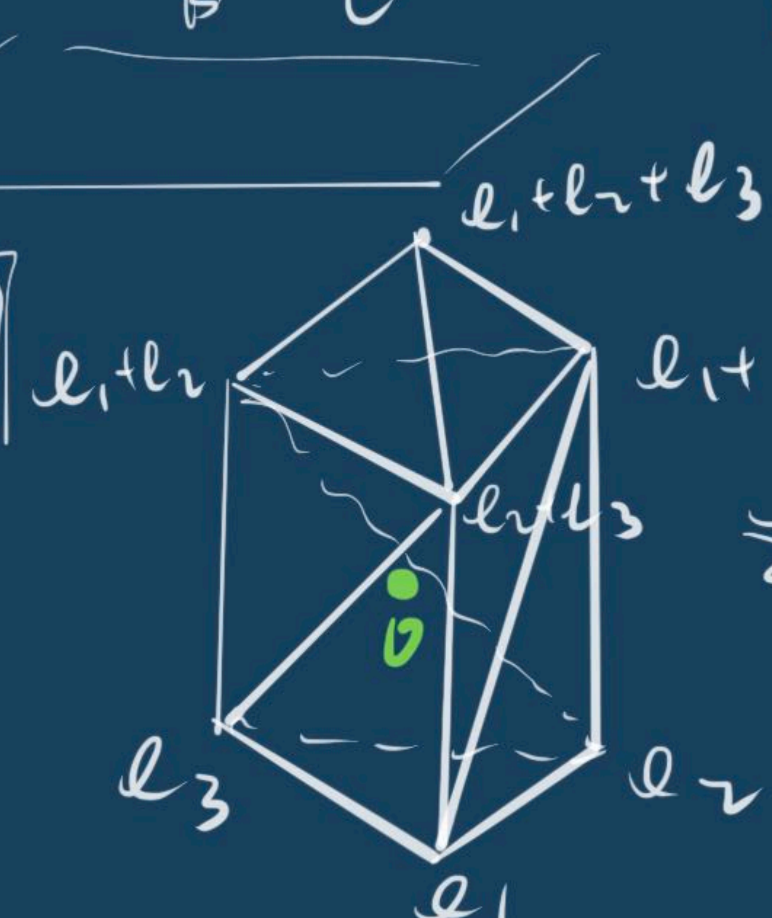
C does not admit any str. convex function

$ht(A') = ht(B') = ht(C')$

$\Downarrow$

$ht(A) > ht(B) > ht(C) > ht(A)$

e.g.



$\Sigma \rightsquigarrow X_\Sigma$ complete sm.

Thm 1 (Schumacher–Tsuji, Ann of Math 04')

(X, H) pol. var.

$\rightsquigarrow X \hookrightarrow \mathbb{P}^N$

deform. $\rightsquigarrow \mathfrak{M}(X,H) \subseteq \mathrm{Hilb}(\mathbb{P}^N)$

Hilb. poly: $\chi(X, \mathcal{O}(mH))$

$\Rightarrow \mathfrak{M}(X,H) / PGL(N+1)$ is always q-proj

Thm 2 (Kollar, Ann of Math 06')

$\forall$ sm toric var can be realized as the moduli of proj/polar var's.

by $\boxed{\text{e.g.}}$, we do have sm, nonproj toric var.

Cor. Thm 1 is wrong.

Rmk. Thm 1 is good if K_X nef.

Rmk: Chow lemma.

$\forall X_\Sigma$ complete

$\exists$ a bir. toric morph. $X_{\tilde{\Sigma}} \to X_\Sigma$ s.t.

$X_{\tilde{\Sigma}}$ is proj.

in dim 2.

Kodaira: proj = dim2 + complete + sm → sm + toric

$\boxed{= \text{still true}}$ ($*$).

pf: of ($*$)

move a small distance to a $\mathbb{Q}$-pt

$\rightsquigarrow$ $s_1, \cdots, s_r$ on $\rho_1, \cdots \rho_r$ $\mathbb{Q}$-pts.

$\rho_{r+1} = \rho_1$

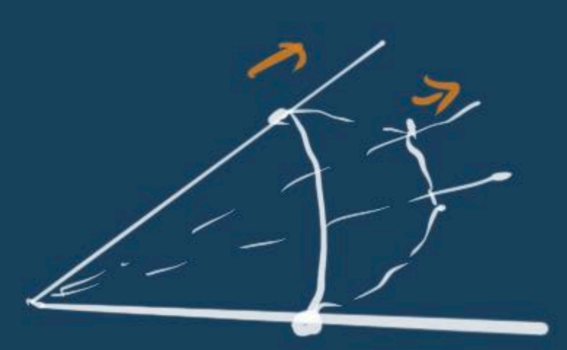
σ_i :

def: $F|_{s_i s_{i+1}} = 1$, extend to be a pw linear func.

multiply by m, $m >> 0$, $m \in \mathbb{N}$.

mF is the str. conv. pw linear function we want. $\Rightarrow$ proj.

Rmk: in dim 3, this doesn't work.



moving doesn't work.