Proj toric var's. $/k=\bar{k}$

X is the affine cone / Y.

Recall: $\mathbb{N}$-graded $R = \bigoplus_{d\geq 0} R_d$ (gen by R_1)

$R_0 = k$

$$\begin{array}{ccc} R & & \\ \uparrow & & \text{deg 1: } f_i \\ k[x_0,\dots,x_N] & & \updownarrow \\ \uparrow & & x_i \\ I & & \\ \uparrow & & \\ 0 & & \end{array}$$

defining eqns (homog. ideal)

$X = \operatorname{Spec} R \qquad Y = \operatorname{Proj} R$

$$\begin{array}{ccc} X & & Y \\ \downarrow & & \boxed{?} \downarrow \\ \mathbb{A}^{N+1}_{x_0,\dots,x_N} & & \mathbb{P}^N \end{array}$$

irr. ideal $R_+ = \langle f_0, \dots, f_N \rangle$

Y is covered by $N+1$ affine pieces

$\bigcup_{j=0}^{N} Y^{(j)}$, $Y^{(j)} = \operatorname{Spec} R[\frac{1}{f_j}]_0$ ← take deg 0.

(dehomogenization)

locally: $R[\frac{1}{f_j}] \twoheadleftarrow k[u_0/u_j, \dots, \widehat{u_j/u_j}, \dots u_N/u_j]$ ← remove

More general: $R = \bigoplus_{d\geq 0} R_d$ f.g. $\mathbb{N}$-grade k-alg. no nilp.

$\rightsquigarrow$ $X = \operatorname{Spec} R$ irr. ideal. $R_+ = \bigoplus_{d\geq 1} R_d$

$Y = \operatorname{Proj} R$

$\bigcup_{j=0}^{N} D_+(f_j)$, $D_+(f_j) = \operatorname{Spec} R[\frac{1}{f_j}]_0$

for $f_0, \dots, f_N$ might be sing.

deg: $d_0, \dots, d_N$ ✓

$\rightsquigarrow X \xrightarrow{F} \mathbb{P}^N(d_1, \dots, d_N)$ wted proj space

$(= \mathbb{A}^{N+1} - \{0\} / k^*$: action.

$t \cdot (x_0, \dots, x_N) = (t^{d_0} x_0, \dots, t^{d_N} x_N)$

$R \xleftarrow{F^*} k[x_0, \dots, x_N]$

$f_i \leftarrow x_i$

$\rightsquigarrow$ polarized var.

i.e. a var. + ample line bundle

$(\operatorname{Proj} R, \mathcal{O}_Y(1))$

~~locally~~ : $H^0(D_+(f), \mathcal{O}(1)) = R[\frac{1}{f}]_1$ ← deg 1.

What did we do?

~~Start~~ from $f_0, \dots, f_N \in R$, deg 1

$\forall g \in R_+$, $g^m \in \langle f_0, \dots, f_N \rangle$

($\Leftrightarrow$ $Y = \operatorname{Proj} R$ covered by $D_+(f_j)$)

$\Rightarrow$ $\begin{cases} Y = \operatorname{Proj} R \xrightarrow{F} \mathbb{P}^N \\ \mathcal{O}_Y(1) \text{ is invertible line bundle. ample.} \\ \mathcal{O}_Y(1)\big|_{D_+(f_i)} \cong \mathcal{O}\big|_{D_+(f_i)} \end{cases}$

Toric case.

Start from a polytope $Q \subseteq M_{\mathbb{R}}$

$\mathrm{Vert}(Q) \subseteq M$

$\rightsquigarrow \widetilde{M} = \mathbb{Z} \oplus M$

$\sigma^\vee = \mathrm{Cone}(1, Q)$

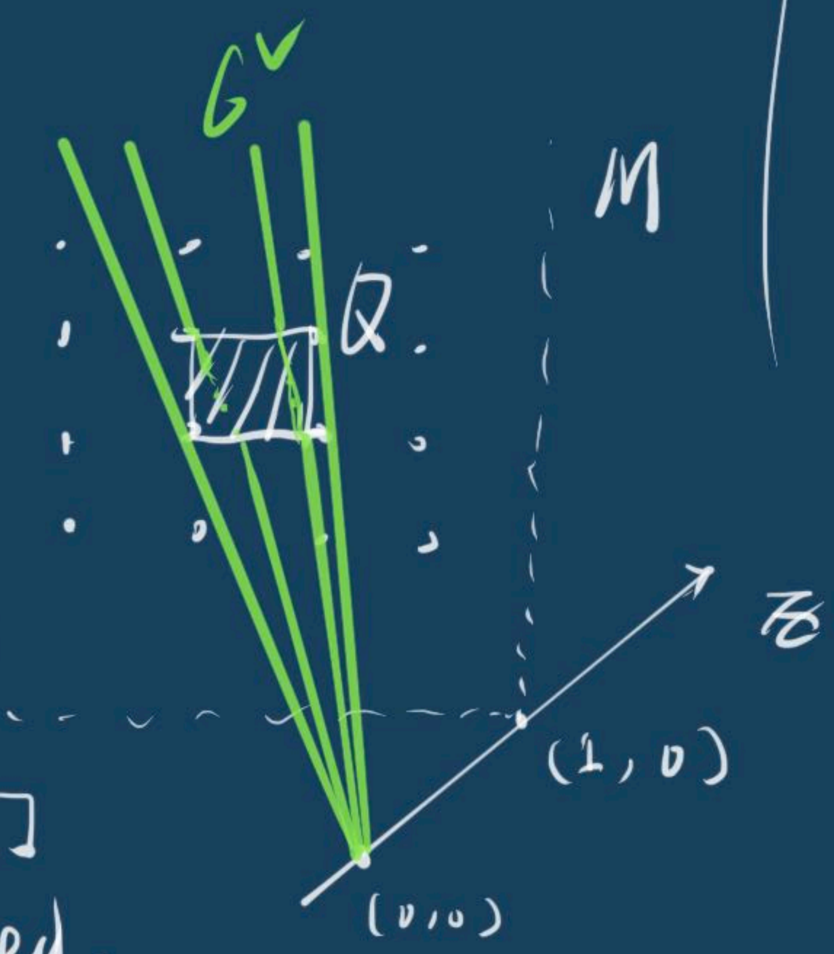
$S = \sigma^\vee \cap \widetilde{M}$

$\Downarrow$

Affine toric var. $\mathrm{Spec}\, k[S]$

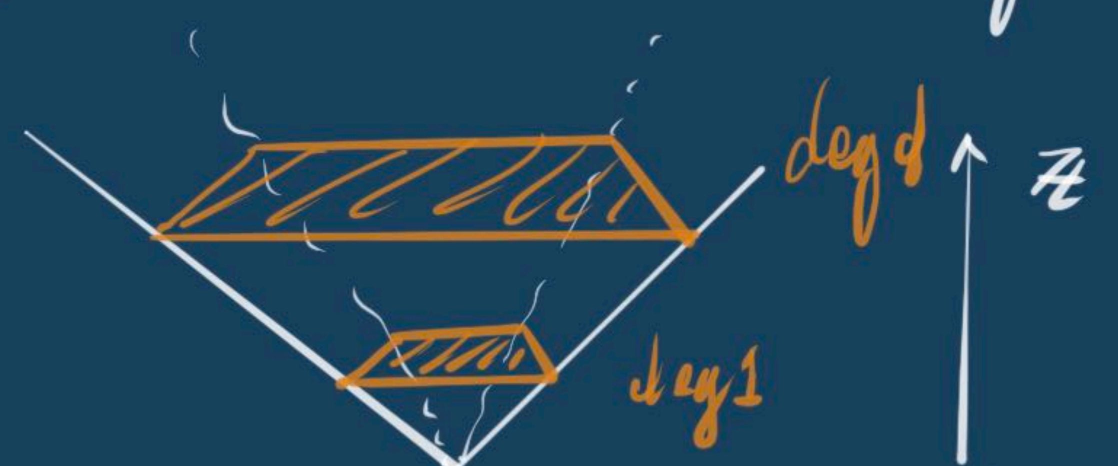
Proj var $\mathrm{Proj}\, k[S]$

$\mathbb{Z}$-graded.



$G(1)$: monomials in deg 1: $\chi^{(1,m)}$, $m \in Q \cap M$

$G(d)$: monomials in deg d: $\chi^{(d,m)}$, $m \in dQ \cap M$



What can we say for $Y = \mathrm{Proj}\, k[S]$?

look at $v \in \mathrm{Vert}(Q) \rightsquigarrow \mathcal{U}_v = \{\chi^v \neq 0\} = D_+(\chi^v)$

$\qquad\qquad\qquad\qquad \shortparallel$

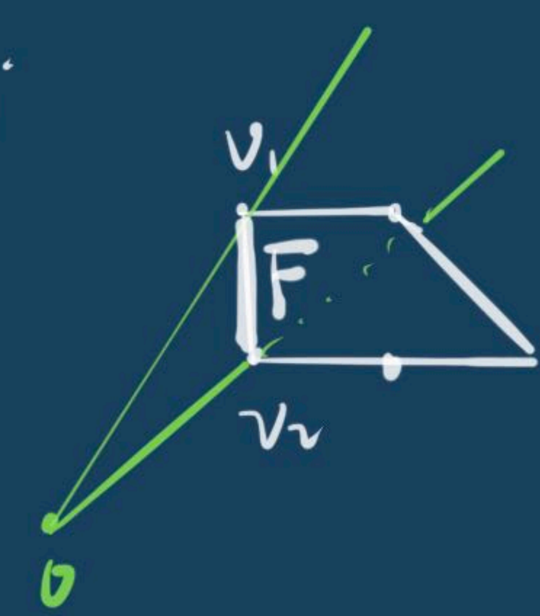
$\qquad \mathrm{Spec}\, k[S][\frac{1}{\chi^{(1,v)}}]_0$

Previously: know $\mathcal{U}_v$ is an affine toric var.

$v \in \mathrm{Vert}(Q)$ 0-faces

for higher dim faces of Q, do the same.

e.g.



$D_+(\chi^{v_1}) \cap D_+(\chi^{v_2})$

$\shortparallel$

$D_+(\chi^{v_1+v_2})$

$\Downarrow$

$\mathcal{U}_{v_1} \cap \mathcal{U}_{v_2} = \mathcal{U}_F$

. all toric affine

. $T \curvearrowright \mathcal{U}_{v_1}$, $\mathcal{U}_{v_2}$ and $\mathcal{U}_F$. compatibly

$T \curvearrowright \mathcal{U}_{v_1} \cup \mathcal{U}_{v_2}$ / glueing along $\mathcal{U}_F$

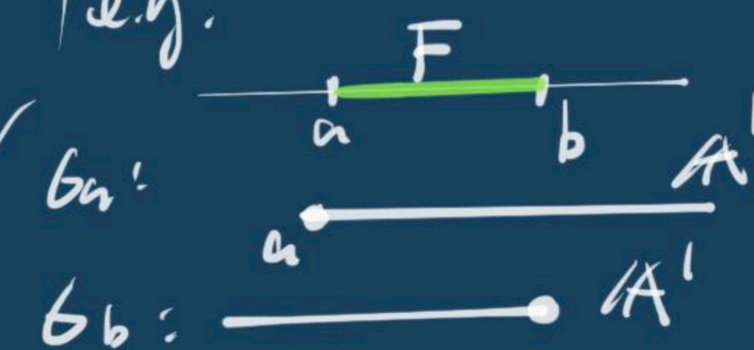
(in general: $\mathcal{U}_{F_1} \cap \mathcal{U}_{F_2}$

$\shortparallel$

$\mathcal{U}_{F_{12}}$ $\quad F_{12}$: smallest face $\supseteq F_1, F_2$)

$\Rightarrow \mathrm{Proj}\, R = Y$ is a toric var.

e.g.



$\Rightarrow \mathbb{P}^1, \mathcal{O}(b-a)$.

glueing not clear.

move to $N_{\mathbb{R}}$.

Again, give a lattice polytope $Q \subseteq M_{\mathbb{R}}$

for each $v \in \mathrm{Vert}(Q)$
$\exists$ cone $\sigma_v^\vee$

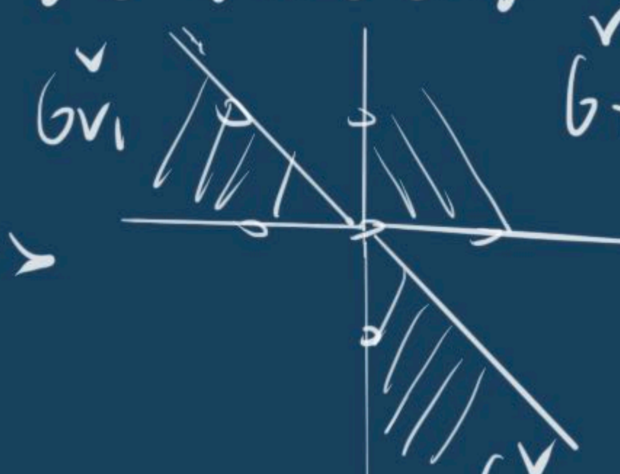
move $\sigma_v^\vee$ to the origin.

$\rightsquigarrow$ $\sigma_v \cap N_{\mathbb{R}}$

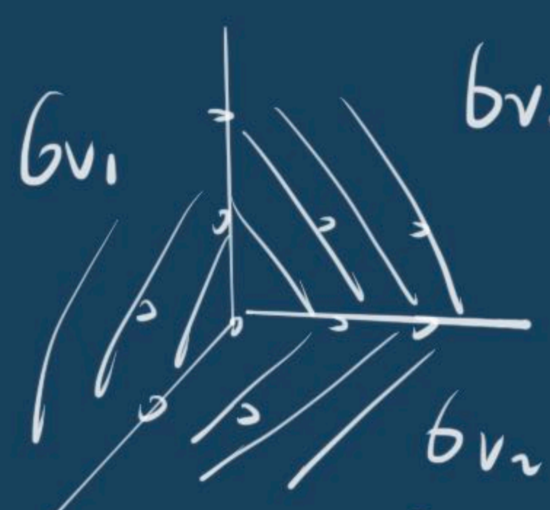
Do this for all $v \in \mathrm{Vert}(Q)$

e.g.

$\rightsquigarrow$

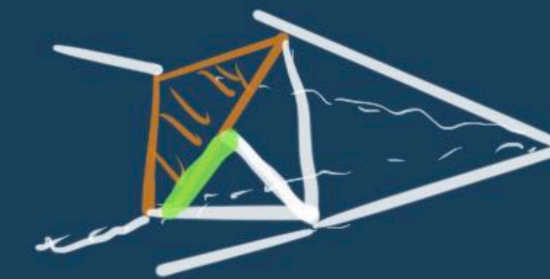


dual $\rightsquigarrow$



note that $\longleftarrow$ also

Thm. $N_{\mathbb{R}} = \bigcup_{v_i \in \mathrm{Vert}(Q)} \sigma_i \quad (\sigma_i = \sigma_{v_i})$ is face matching.
$= \bigsqcup_{F < Q} \sigma_F^{\circ}$



is not a face of

Pf: def $\sigma_i' = \{ n \in N_{\mathbb{R}} \mid \langle n, - \rangle$ achieve it min. @ $v_i \}$

Claim: $\sigma_i' = \sigma_i$

$n \in \sigma_i' \Leftrightarrow \forall u \in Q, \ \langle n, u \rangle \geq \langle n, v_i \rangle$
$\Leftrightarrow \forall u' = u - v_i \ \ \langle n, u' \rangle \geq 0$
$\Leftrightarrow \forall u' \in \mathbb{R}_{\geq 0}(Q - v_i), \langle n, u' \rangle \geq 0$
$\Leftrightarrow \forall u' \in \sigma_i^\vee \ \langle n, u' \rangle \geq 0$
$\Leftrightarrow n \in \sigma_i$

$\forall p \in N_{\mathbb{R}}$, p as a linear functional in $M_{\mathbb{R}}$ ($M_{\mathbb{R}}|_Q$) has its min on the boundary of Q say $F \subseteq Q$

$\Rightarrow p \in \sigma_F = \bigcap_{v_i \in F} \sigma_i \Rightarrow N_{\mathbb{R}} \subseteq \bigcup_{i \in \mathrm{Vert}(Q)} \sigma_i$
" $\supseteq$ " obvious

Face matching: $\sigma_{F_1} \cap \sigma_{F_2} = \sigma_{F_3}$
take F_3 : smallest face $\supset F_1, F_2$

Def: $\Sigma_Q = \{ \sigma_F \subseteq N_{\mathbb{R}} \}$ is called the normal fan of Q.

- $Q_1, Q_2 \subseteq M_{\mathbb{R}}$. $\quad Q_1 \overset{\text{normally}}{\rightsquigarrow} Q_2$ if $\Sigma_{Q_1} = \Sigma_{Q_2}$

$Q_1 : (Y_1, L_1) \quad L_1$
$Q_2 : (Y_2, L_2) \quad \cancel{\cong} \ L_2$
$Y_1 \cong Y_2$

e.g.

same normal fan.

$Q \subseteq M_{\mathbb{R}} \rightsquigarrow (X, L)$

lattice

$\leftarrow$ What can we say for L?

Rmk: polytope $\rightsquigarrow$ normal fan

polytope $\longleftarrow$ normal fan

not always true

Thm. $Q \rightsquigarrow (X, L)$

$$H^0(X, L) = \bigoplus_{m \in M \cap Q} k x^{(1,m)} \quad \left(\Rightarrow h^0(X,L) = \#\{Q \cap M\}\right)$$

e.g.

$$(\mathbb{P}^2, \mathcal{O}(2)) = \mathrm{Proj}\, k[x_0^2, x_1^2, x_2^2, x_0x_1, x_1x_2, x_0x_2]$$

$$h^0(\mathcal{O}(2)) = 6$$

"pf": $T \subset X$ $\quad L$: ample on X $\quad L|_T \cong \mathcal{O}_T$

$$T = \mathrm{Spec}\, k[M] = \mathrm{Spec}\, k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

$$H^0(X, L) \hookrightarrow H^0(T, \mathcal{O}_T) = \left\{\bigoplus_{m \in M} k x^{(1,m)}\right\}$$

$$= \left\{\bigoplus_{m \in M} k x^{(1,m)} \;\middle|\; \mathrm{div}(x^{(1,m)}) \geq 0\right\}$$

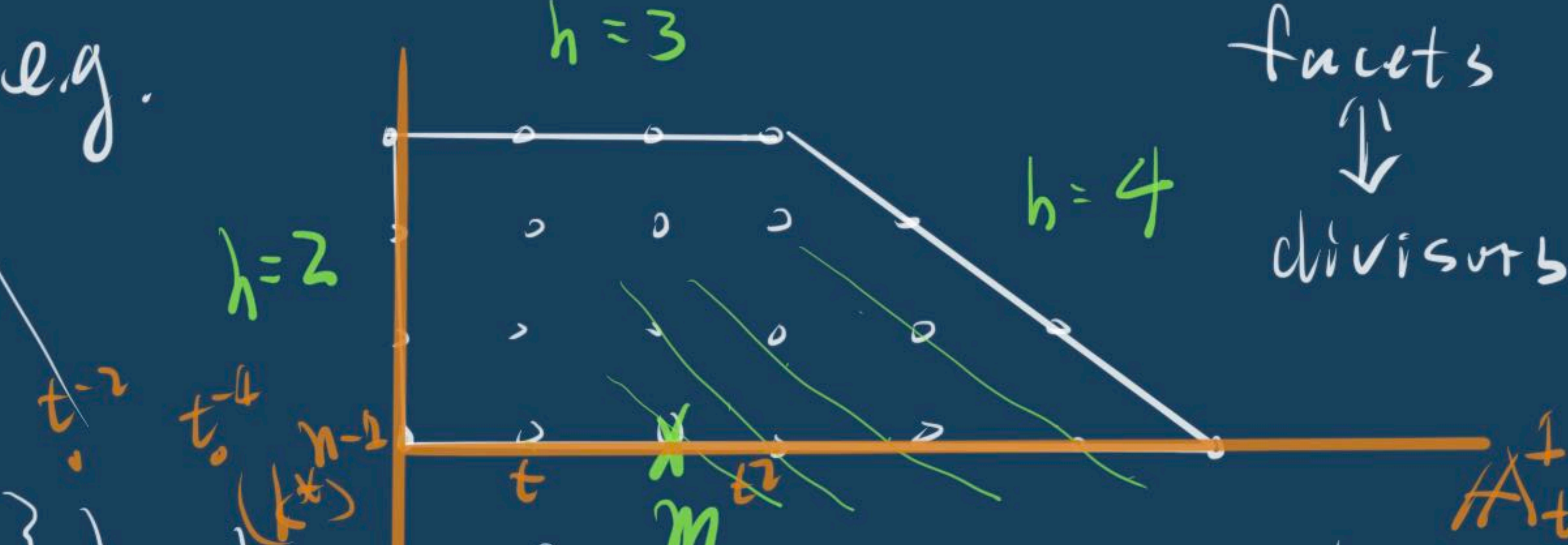
Q: What is $\mathrm{div}(x^{(1,m)})$?

A: $$\mathrm{div}(x^{(1,m)}) = \sum_{\substack{F < Q \\ \text{facets (codim 1)}}} h_F(m) \cdot D_F$$

$h_F(m)$: distance from m to F

D_F: divisor corresp. to F

e.g.



facets $\Updownarrow$ divisors.

coef. of $x^{(1,m)}$ wrt D_F

$F \rightsquigarrow U_{\sigma_F} = \underbrace{(k^*)^{n-1}} \times \mathbb{A}^1$

problem is reduced to $\mathbb{A}^1_t$ | no valuation

$\mathrm{div}(x^{(1,m)}) = \mathrm{div}(t^h) = h$ along F (D_F)

in Q: regular

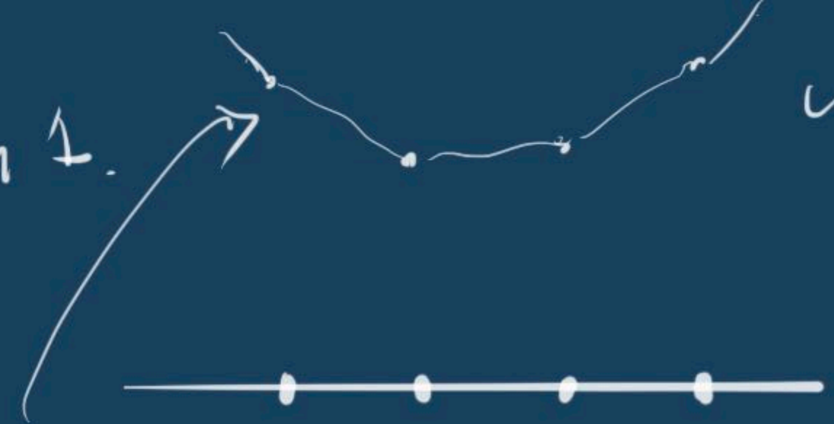
out of Q: poles.

Rmk:

fan Σ $\exists$ polytope Q s.t. $\Sigma = \Sigma_Q$

$(1-\lambda) f(x_1) + \lambda f(x_2) \geq f((1-\lambda)x_1 + \lambda(x_2))$

convexity of Q $\longleftrightarrow$ convexity of the piecewisely linear function on Σ.

dim 1. 

convex function.

slopes diff. $\Leftrightarrow$ strongly convex.

Rmk: • $Q \to (X, L)$

$nQ \to (X, L^{\otimes n})$ Veronese emb.

• $Q_1, Q_2 \rightsquigarrow (X_1, L_1), (X_2, L_2)$

$\Downarrow$

$Q_1 \times Q_2 \rightsquigarrow (X_1 \times X_2,\ p_1^*L_1 \otimes p_2^*L_2)$ Segre emb.

Now: $(X, \underset{\text{ample.}}{L})$ from Q $\rightsquigarrow \varphi_L : X \to \mathbb{P}^N$

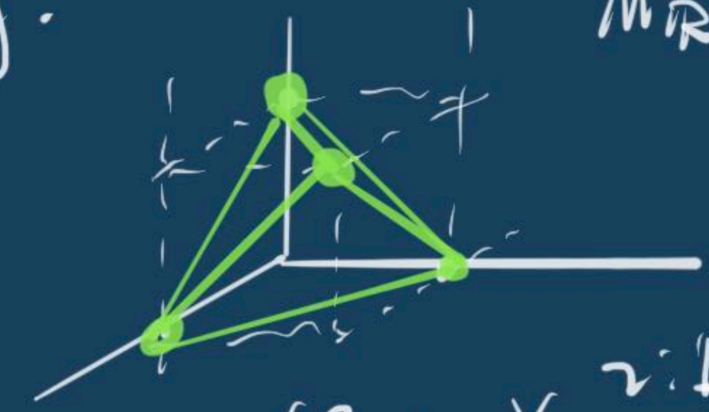
What can we say for φ_L?

non toric e.g. C_g curve, $g \geq 2$

$D = p = \{pt\}$. $\mathcal{O}(D)$ is ample not very ample.

$\mathcal{O}((2g+1)D)$ is very ample.

Toric e.g.



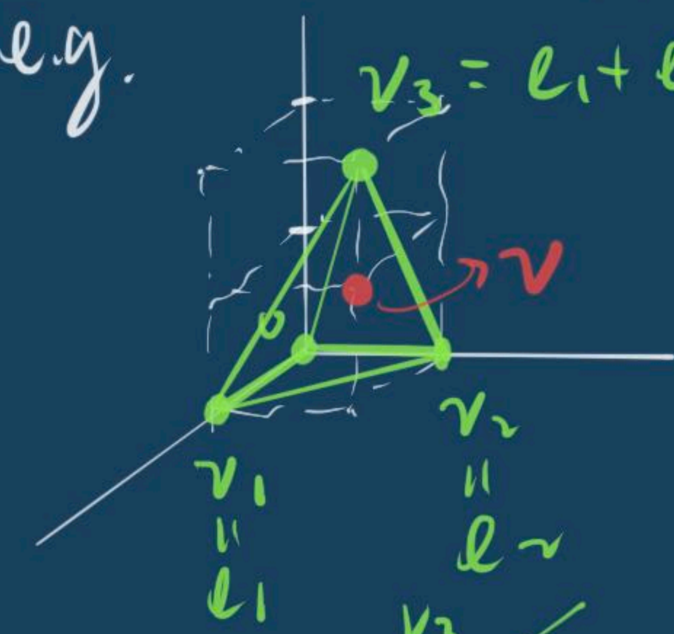
Q gen a sublatti[ce] $M' \subseteq M$

$[M : M'] = 2$

$\varphi_L : X \xrightarrow{2:1} \mathbb{P}^3$

not emb.

More e.g.



$Q \not\ni v = e_1 + e_2 + e_3$

$= \frac{1}{3} \cdot 2v_1 + \frac{1}{3} \cdot 2v_2 + \frac{1}{6} \cdot 2 \cdot v_3$

$\in 2Q$.

L: (X, L) ample

$\downarrow$ not very ample

$(X, L^{\otimes 2})$ $L^{\otimes 2}$ very ample.

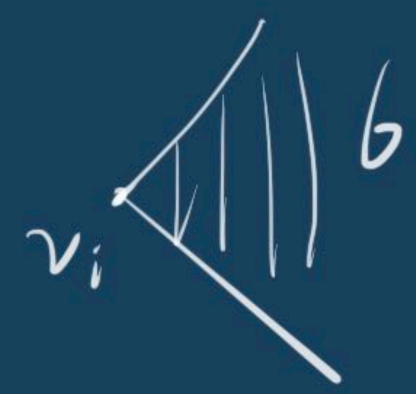
Thm. $(X, \underset{\text{ample.}}{L}) \longleftarrow Q$ $\dim X = n$

$\Rightarrow (n-1)L$ very ample.

Q: What is very ample on a polytope Q?

A: ample +

$\forall v \in \mathrm{Vert}(Q)$ $\{m - v \mid m \in Q \cap M\}$ gen. M. (*)



$\mathbb{P}^{h_0-1}$ $\quad h_0 = \#\{Q \cap M\}$

$\bigcup_{i=0}^{h_0} \mathbb{A}_i^{h_0-1}$, $\underline{\mathbb{A}_i^{h_0-1} = \{x_i \neq 0\}}$

$\forall$ full dim $\sigma \in \Sigma_Q \rightsquigarrow U_\sigma = \mathrm{Spec}\, k[\sigma^\vee \cap M]$

$k[\sigma^\vee \cap M] \longleftarrow \Leftarrow k[\mathbb{N}^{h_0-1}]$ $\quad m - v_i \leftarrow\!\shortmid 1_m$

$\Leftrightarrow$ *

$\Updownarrow \quad \Updownarrow$

$U_\sigma \longrightarrow \hookrightarrow \mathbb{A}_i^{h_0-1}$

want! to be.

Rmk: X sm. $\dim X = n$. L ample.

$\cancel{k = \bar{k}}$

$\mathrm{char}(k) = 0$

Fujita conjecture . $K_X + (n+1)L$ BPF

. $K_X + (n+2)L$ very ample

True for toric.

Rmk: (X, L), L very ample.

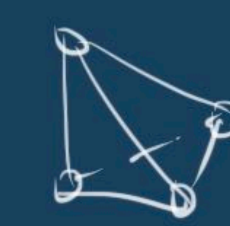
$\varphi_L : X \hookrightarrow \mathbb{P}^N$

Q: What is $\deg(X)$?

A: $\deg(X) =$ normalized vol.

$=$ Euclid vol. $\cdot \dim(X)!$

e.g. $(\mathbb{P}^3, \mathcal{O}(1))$



$\deg. = (\frac{1}{2} \times 1 \times \frac{1}{3}) \cdot 3!$

$= 1$

$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$

$\deg. = 1 \times 2! = 2.$

Rmk: very ample $=$ ample $+$ (*)

$\Downarrow$

ample $=$ strongly convex pw. linear fun.

$\Downarrow$

semiample : D, $|mD|$ BPF

$\Downarrow$ $\Uparrow$ toric $=$ convex pw. linear

nef : $D \cdot C \geq 0$ $\forall$ curve $C \subseteq X$.

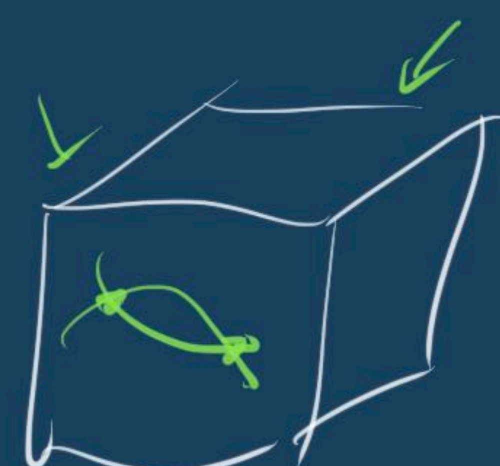
In DG: $\exists$ Whitney emb thm.

$$\begin{array}{c} M \text{ sm. } \mathbb{R}\text{-mfd } \dim n \\ \downarrow \\ \mathbb{R}^{2n} \end{array}$$

In AG: we lose this

- X is sm. proj $\dim X = n$

  $i: X \hookrightarrow \mathbb{P}^{2n+1}$



- X is sm. complete, (compact)

  i may not exist. e.g. Hironaka's example.

- In dim 2. complete + sm = proj. not true ≥ 3.

- If "sm" is removed.

  $\rightsquigarrow$ $X \hookrightarrow \mathbb{P}^N$ DNE example from toric var's.

  need Σ fan. does not admit a strongly convex pw. linear fun.

- If nilp's are allowed $\rightsquigarrow$ strange:

$X := \operatorname{Spec} k[x_0, \cdots, x_n] / \langle x_0^2, \cdots, x_n^2 \rangle \hookrightarrow \mathbb{P}^N$. $N \geq n$

but $\dim_k X = 0$.

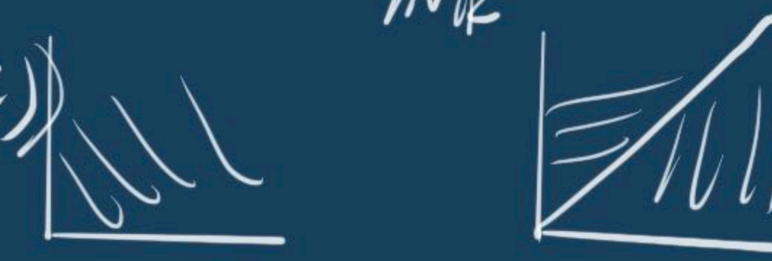
e.g.

$E \cong \mathbb{P}^1$

$(\mathrm{Bl}_1 \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2) \otimes \pi^*(-E))$

last time

$N_{\mathbb{R}}$



$\mathbb{A}^2 \leftarrow \mathrm{Bl}_0 \mathbb{A}^2$

today:

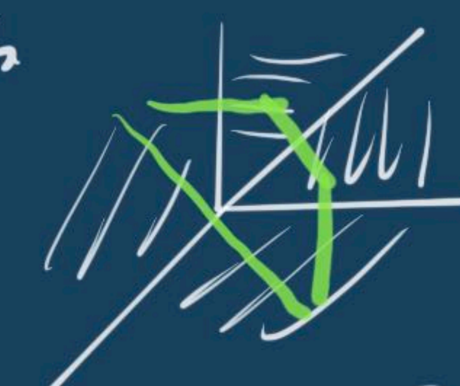
$(\mathbb{P}^2, \mathcal{O}(m))$

$\rightsquigarrow$

σ_1, σ_2, σ_3 $\mathbb{P}^2_{x,y,z}$

$\sigma_1 \longmapsto \mathbb{A}^2_{x,y}$

thus



π

$\mathrm{Bl}_1 \mathbb{P}^2$
$1 = (0,0,1)$

e.g.

$\mathbb{P}^1$ fibers

length $r+1$

Hirzebruch surf. $\mathbb{F}_r$

$\mathbb{P}^1$-fibration

$\mathbb{P}^1$

S_0, $S_0^2 = -r$

S_∞, $S_\infty^2 = r$.