Last time: not saturated.

⇓

vert: cuspital curve 
not normal.

expect: S affine semigp

$\operatorname{Spec} k[S]$ given by the cone gen. by S

Def: $\underbrace{S}_{\text{aff. semigp}} \subseteq M$ is saturated if

$\forall\, m \in \mathbb{Z}S$, $km \in S \Rightarrow m \in S$ for $k \in \mathbb{Z}_+$.

Thm: S saturated $\Leftrightarrow$ $X = \operatorname{Spec} k[S]$ normal.

---

non. e.g.

two surf's.

is R_1, not S_2

Rmk: normal: alg: loc. integral closed

geo: $R_1 + S_2$

non normal ($\Leftarrow$ sing. in dim 1)

- not R_1 (reg. in codim)

codim(sing, X) = 1

dim X = 2

⇓

$X = V\{y^2 = x^3\} \times \mathbb{A}^1_z$ X: not R_1.

- S_2: depth $\mathcal{O}_{X,p} \geq \min\{2, \dim \mathcal{O}_{X,p}\}$

S_i, $i = n$, S_n is called Cohen-Macaulay

X is S_2, S_2 at all $p \in X$

For some people, S_i "= dim by cutting"

| A M: A-mod
| $x_1, \dots, x_n \in A$
| s.t. $M/(x_1, \dots, x_{i-1})$
| x_i is not a 0-div of $\overline{M}$

Krull: "CW" version

⇓ ↗

Serre:

geo of S_2:

- X, $Y \subseteq X$, codim$(Y, X) \geq 2$

remove Y, $X \setminus Y$

$\rightsquigarrow$ $\exists!$ recover X

- locally connectedness is not change by removing codim ≥ 2 subvar.

Pf: (of the thm)

"$\Leftarrow$" $k[S]$ int. closed, $ks \in S$, $k \in \mathbb{Z}_+$

$$\begin{array}{ccc} x^{ks} & x^{s} & \text{in } k[S] \\ \| & \| & \\ a & b & \end{array}$$

$b^k = a \Rightarrow b$ is a root of $x^k - a =$

$\overset{\text{int. closed}}{\Rightarrow}$ $b \in k[S] \Rightarrow s \in S$, i.e. S is sat.

"$\Rightarrow$" S: affine sat. $k[S] \subset k[M]$

Recall: alg. facts:

1) A int closed.
$\Rightarrow$ $\forall$ locallization $S^{-1}A$ is also int. closed

2) A_1, A_2 int. closed $\operatorname{frac}(A_1) = \operatorname{frac}(A_2)$
$\Rightarrow$ $A_1 \cap A_2$ int closed.

$$\begin{array}{c} S \\ \updownarrow \\ \sigma^\vee \text{ in } M_{\mathbb{R}} \\ \| \\ \cap \text{ half spaces} \end{array}$$

int. closed by 2)

$$k[S] = \cap \underbrace{k[S]_{f_i}}_{\text{coord. changing}}$$

$$k[S]_{f_i} \cong k[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}] = k[x_1, \dots, x_n]_{x_2 \cdots x_n}$$

is int. closed by 1)

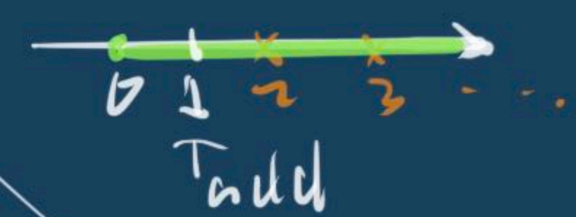
Summary:

$$\{\text{normal affine toric var's}\} \overset{1:1}{\hookrightarrow} \{\text{strongly convex cones in } N_{\mathbb{R}}\}$$

$$\updownarrow 1:1 \qquad\qquad \overset{1:1}{\nearrow} \quad \uparrow \text{ Gordan's lemma.}$$

$$\{\text{saturated affine semigp}\}$$

Normalizations:



$\rightsquigarrow$ $\operatorname{Spec} k[x,y]/(y^2 - x^3)$
$\|$
$\operatorname{Spec} k[t^2, t^3]$
$\uparrow$
$\operatorname{Spec} k[t]$

In general: S generates $\sigma^\vee \subseteq M_{\mathbb{R}}$

$\rightsquigarrow$ $S \hookrightarrow \sigma^\vee \cap M$

$\Downarrow$

$k[S] \hookrightarrow k[\sigma^\vee \cap M]$

$\Downarrow$

normalization: $\operatorname{Spec} k[\sigma^\vee \cap M] \to \operatorname{Spec} k[S]$

(e.g. $S = \langle s, st, t^2 \rangle$

$\Downarrow$

$\sigma^\vee \cap M = \langle s, t \rangle$

$k[S] \hookrightarrow k[s,t]$

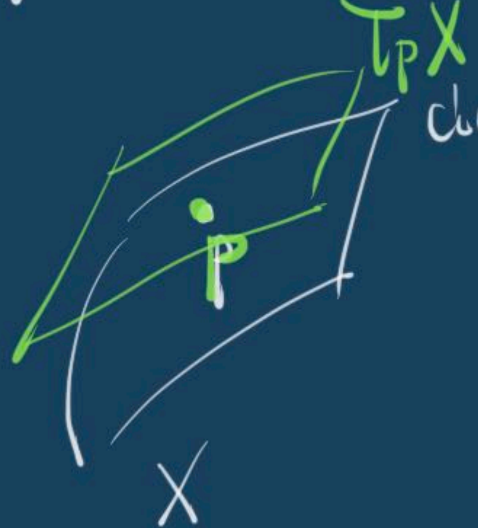
$\|$

$k[s, st, t^2]$

take spec: $\mathbb{A}^2_{s,t} \to \operatorname{Spec} k[S]$

Singularities (on affine toric var's)

Recall: (co)tangent space in Zariski sense.



closed pt $p \in X \Leftrightarrow m_p$ maximal ideal

e.g. $p = (a,b) \in \mathbb{A}^2_{x,y}$

$\Updownarrow$

$m_p = \langle x-a, y-b \rangle$

$\shortparallel$

all regular functions vanishing @ $(a,b) = p$

$\Rightarrow \; m_p^2 =$ regular function vanishing at p deg ≥ 2.

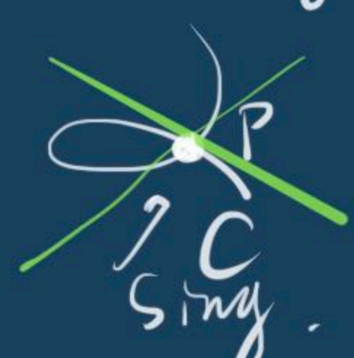
$\Rightarrow \; m_p / m_p^2 =$ deg 1 $=$ linear regular functions vanishing @ p

$= T_p^* X \quad \left(T_pX = (m_p/m_p^2)^\vee \right)$

X is smooth at $p \Leftrightarrow \dim T_pX = \dim_p T_p^* X \overset{\text{need}}{=} \dim X$

X is sing. $\cdot\cdot \Leftrightarrow$ 〃 〃 $> \dim X$

e.g.



$\dim T_p C = 2 > \dim C = 1.$

For toric var's.

$T \curvearrowright U_\sigma = \operatorname{Spec} k[S_\sigma]$

$S_\sigma = \sigma^\vee \cap M$ for some σ: strongly convex polyhedral rational cone.

$p \in \operatorname{Spec} k[S_\sigma]$

$\Updownarrow$

$\gamma_p : k[S] \to \mathbb{C}$ or just $S \to \mathbb{C}$ (plug in p)

p is a fixed pt $\Leftrightarrow \gamma_p$ is fixed under $T \curvearrowright$

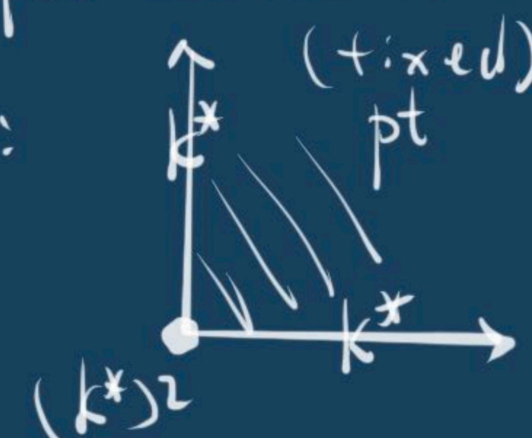
i.e. $\chi^m(t) \cdot \gamma_p(m) = \gamma_p(m)$ $\forall t \in T$, $\forall m \in S$

need: $\gamma_p(m) = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0. \end{cases}$

γ_p unique $\Leftrightarrow p$ unique.

$\operatorname{Spec} k[S_\sigma] \ni 0$ is a fixed pt.

e.g. $N_{\mathbb{R}}$:



$\mathbb{A}^2$

Rmk: if σ is not full dim. $\Rightarrow$ no fixed pt

$p = 0 \in \operatorname{Spec} k[S_\sigma] \longleftrightarrow$ maximal ideal

fixed pt   which one?

$m_p = \langle \chi^m \mid m \in S_\sigma^* \rangle \subseteq k[S_\sigma]$

$S_\sigma^* = S_\sigma \setminus \{0\}$

max. because $k[S_\sigma]/m_p = k$

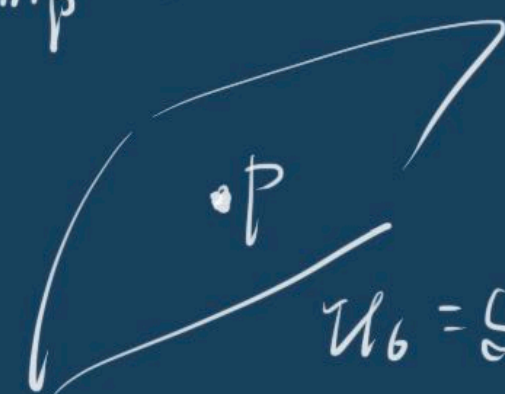
vanishes at 0.

$m_p = \bigoplus_{m \in S^*} k\chi^m$

$m_p^2 = \bigoplus_{\substack{m \in S^* + S^* \\ m \in S^*}} k\chi^m$

$m_p / m_p^2 = \bigoplus_{\substack{m \neq m_1 + m_2 \\ m_1, m_2 \in S^* \\ m \in S^*}} k\chi^m$

(m primitive)



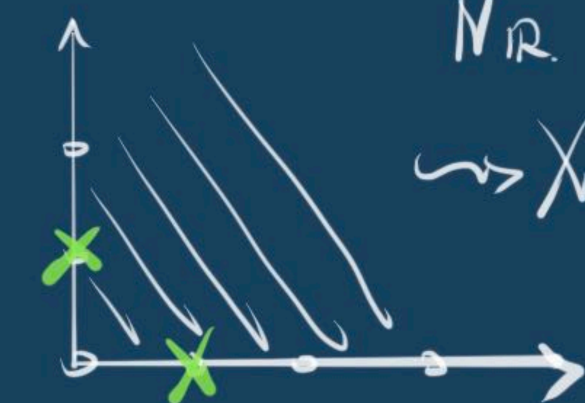
$U_\sigma = \operatorname{Spec} k[S_\sigma]$

smooth at p

$\Updownarrow$

$\dim m_p/m_p^2 = \dim X$

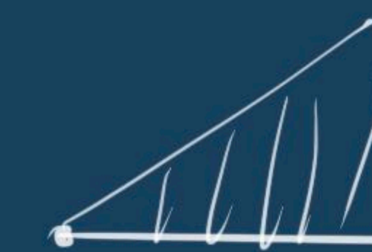
e.g.



$\rightsquigarrow X$ is smooth $X = \mathbb{A}^2$.

1. $\{ \uparrow \rightarrow \}$ is an $\mathbb{R}$-basis of $\mathbb{R}^2$

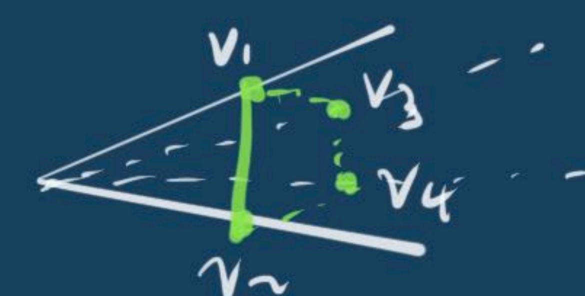
2. $\forall$ pt in N can by written ($\mathbb{Z}$-pt) as $\mathbb{Z}$-comb.'s of $\uparrow$ and $\rightarrow$



e.g. violate 1.

in dim 2, all cones are simplicial

in dim ≥ 3, $\uparrow$ not true.



four gen's for the cone

$\vee$

$3 = \dim X$ (= rank M).

$+\infty$ generators

$\vee$

3

eg. violate 2.

X sing.

$\dim m_p/m_p^2 = 3$

$\vee$

$\dim X = 2$

Thm. $U_\sigma = \operatorname{Spec} k[S_\sigma]$ affine normal

U_σ is smooth $\iff$ 1. $\sigma^\vee$ is simplicial

2. gen's of rays gen M of $\sigma^\vee$ $\iff$ 1. σ simplicial

2. gen's of rays of σ gen. N.

Def: $\sigma^\vee$ is simplicial if its rays from a $\mathbb{R}$-basis of $N_\mathbb{R}$ ($M_\mathbb{R}$)

$\mathbb{Q}$-factorial: $\forall$ Weil div $D \Rightarrow \exists m$, mD Cartier.

Rmk: later glue cones $\rightsquigarrow$ fans $\rightsquigarrow$ nonaffine toric var.

all σ's simplicial

$\Downarrow$

X is $\underline{\mathbb{Q}\text{-factorial}}$

Localizations.

Recall: on $X = \operatorname{Spec} A$, we have 2 main kinds of localizations

1. stalks: $\mathfrak{p} \subseteq A$ prime $\Rightarrow A_{\mathfrak{p}}$

$\boxed{2.}$ principal open subsets $\{\operatorname{Spec} A_f =: D(f)\}$, $f \in A$, form a basis of X

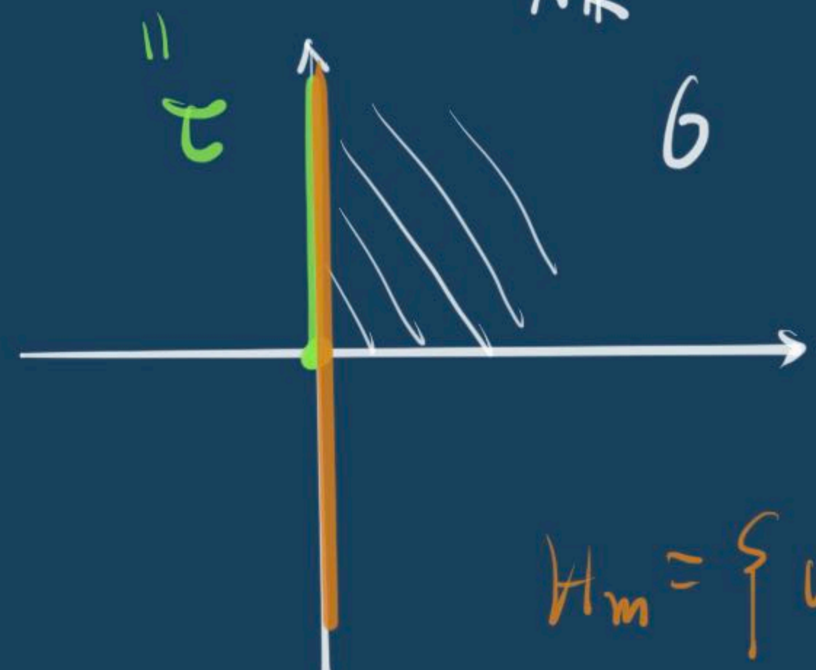
In toric world:

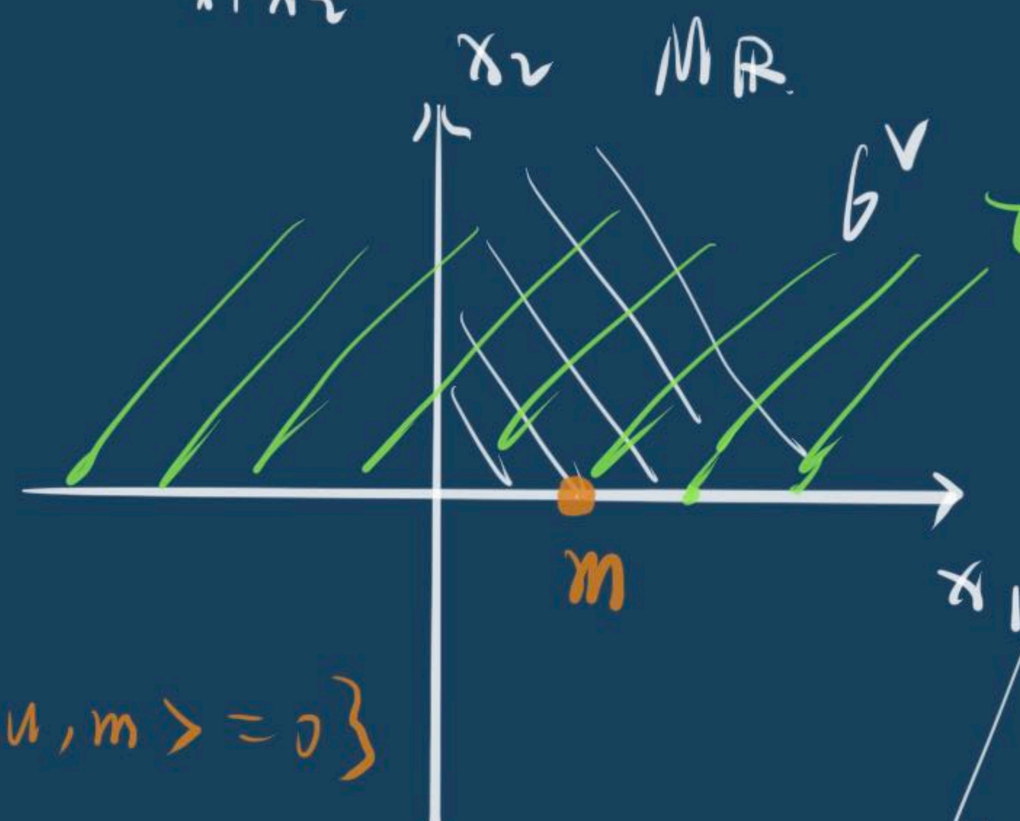
e.g. $X_\tau = \operatorname{Spec} k[x_1, x_2]_{x_1} = \operatorname{Spec} k[x_1^{\pm 1}, x_2] = k^*_{x_1} \times \mathbb{A}^1_{x_2}$

$$\downarrow$$

$Y_\sigma = \operatorname{Spec} k[x_1, x_2] = \mathbb{A}^2_{x_1, x_2}$

$(k^*)^2 \sqcup k^*$





$H_m = \{u \in N_{\mathbb{R}}, \langle u, m\rangle = 0\}$

$\Rightarrow \tau = H_m \cap \sigma$.

In general:

$\tau \leq \sigma \subseteq N_{\mathbb{R}} \rightsquigarrow S_\tau \supseteq S_\sigma$

face; $S_\tau = \tau^\vee \cap M$, $S_\sigma = \sigma^\vee \cap M$

$\rightsquigarrow k[S_\tau] \longleftarrow k[S_\sigma]$

choose $m \in \sigma^\vee \cap M$ s.t. $\underline{\tau = H_m \cap \sigma}$ (*)

Prop. $k[S_\tau] = k[S_\sigma]_{\chi^m}$

$\quad (k[\tau^\vee \cap M])$

pf: $\tau \subseteq H_m \Rightarrow \langle m, u\rangle = 0, \ \forall u \in \tau$

also by def: $\tau^\vee = \{a \in M_{\mathbb{R}}, \langle a, u\rangle \geq 0 \ \forall u \in \tau\}$

$\Rightarrow \pm m \in \tau^\vee$ "="

$\Rightarrow S_\sigma + \mathbb{Z}(-m) \subseteq S_\tau$.

"$\supseteq$" also true.

$\Downarrow$

$S_\tau = S_\sigma + \mathbb{Z}(-m)$

$\Downarrow$

$k[S_\tau] = k[S_\sigma]_{\chi^m}$

Choose a finite set F_σ gen. σ.

$\sigma = \operatorname{Cone}(F_\sigma)$

$\forall m' \in S_\tau$, $C := \max_{u \in F_\sigma} |\langle m', u\rangle|$

$$\langle m' + Cm, u\rangle = \langle m', u\rangle + C\langle m, u\rangle = \begin{cases} 0, & u \in \tau \\ \geq -C + Ck \geq 0 & u \in \sigma \setminus \tau \end{cases}$$

$k \geq 0$, since $u \in \sigma$, $m \in \sigma^\vee$

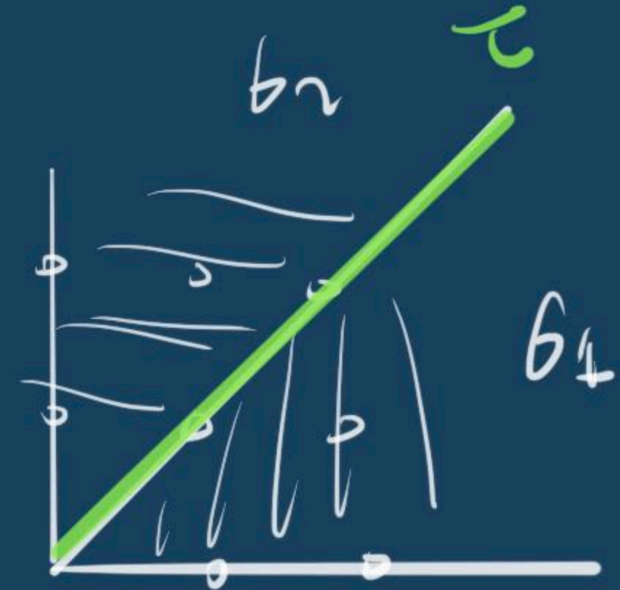
$k \geq 1$, $u \notin \tau$, $m \perp \tau$ by (*)

Cor. $\sigma \cap \sigma' = \tau$

$\operatorname{Spec} k[S_\tau] = U_\tau$ is contained in $U_\sigma, U_{\sigma'}$

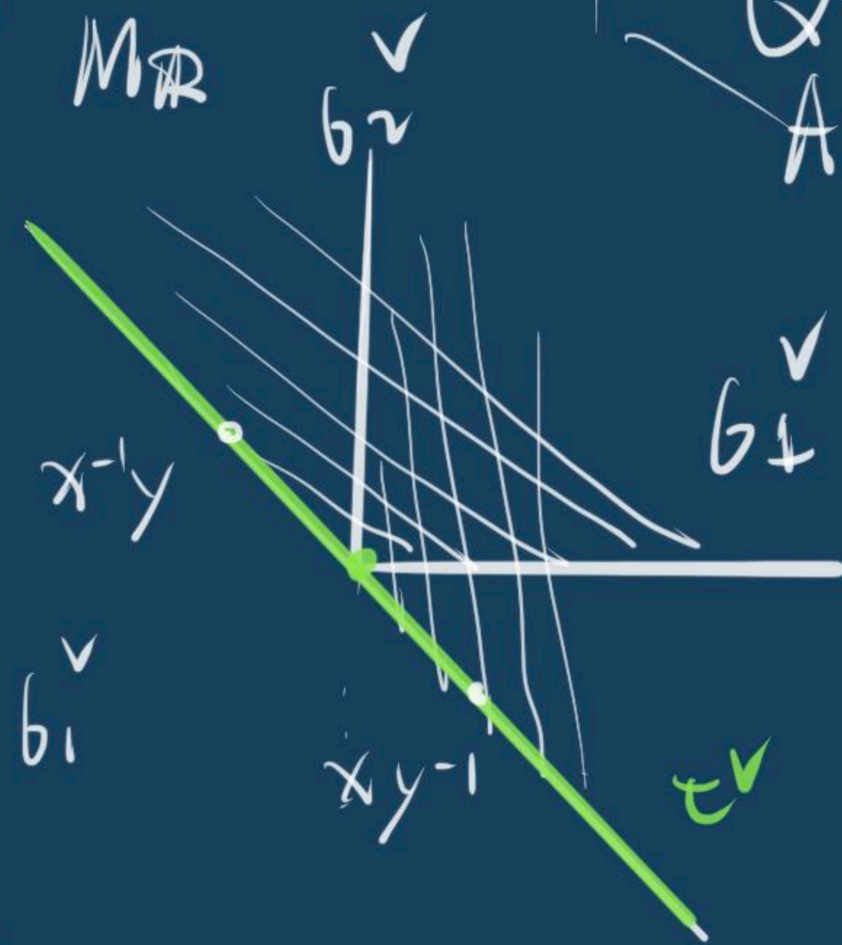
$\rightsquigarrow$ $U_\sigma, U_{\sigma'}$ glue along U_τ.

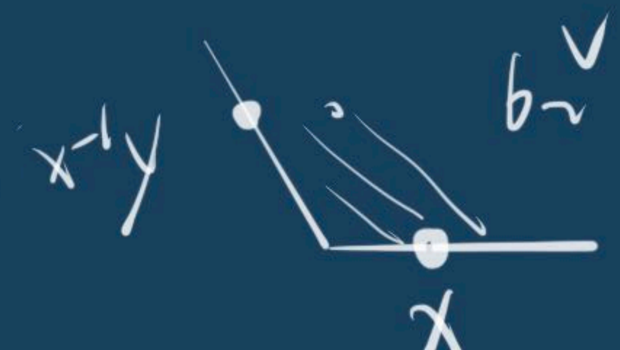
e.g.

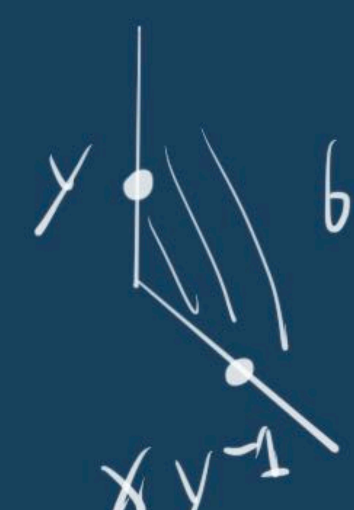


$N_{\mathbb{R}} \overset{\text{dual}}{\longleftrightarrow} M_{\mathbb{R}}$

$\rightsquigarrow X$







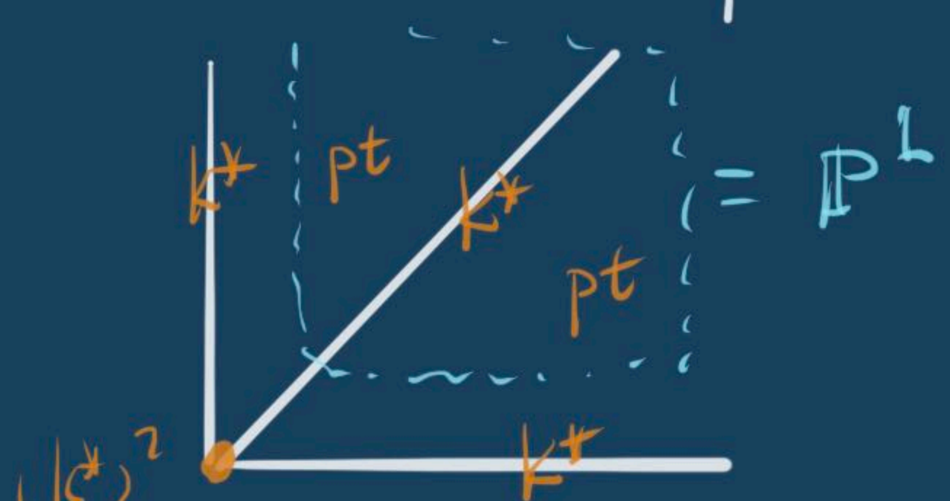
$$\left.\begin{aligned} \mathbb{A}^2 &= U_{\sigma_1} = \operatorname{Spec} k[xy^{-1}, y] \\ \mathbb{A}^2 &= U_{\sigma_2} = \operatorname{Spec} k[x, x^{-1}y] \\ k^* &= U_\tau = \operatorname{Spec} k[\underbrace{x^{-1}y, \ xy^{-1}}_{\text{pairwise inverse.}}] \end{aligned}\right\} \Rightarrow$$

by coordinate change.

$X = Bl_0 \mathbb{A}^2_{x,y}$

$E \cong \mathbb{P}^1$

$E^2 = -1$



Eqn's of ~~toric~~ var's. $S = \langle s_1, \dots, s_n \rangle$

S: affine semigp. $S \hookrightarrow \mathbb{Z}^m$

$\Downarrow$

$\exists$ $\mathbb{N}^n \xrightarrow{\varphi}$ s.t. $\operatorname{Im}(\varphi) = S$

$\rightsquigarrow \Phi : k[t_1, \dots, t_n] \xrightarrow[\text{hom.}]{\text{semigp}} k[S]$

Q: What is $\ker \Phi$?

A: Prop. $\ker \Phi = \langle t^a - t^b \mid \varphi(a) = \varphi(b) \rangle$

Pf: need $\ker \Phi \subseteq \langle t^a - t^b \mid \varphi(a) = \varphi(b) \rangle$

Note: $k[S]$ is $\mathbb{Z}^m$-graded.

$k[\mathbb{N}^n]$ is also $\mathbb{Z}^m$-graded

$\Downarrow$ $t^\lambda, \lambda \in \mathbb{N}^n, \deg(t^\lambda) = \varphi(\lambda)$

$\Downarrow$

Φ is homogeneous

$\Downarrow$

$\ker \Phi$ is homogeneous.

$\Downarrow$

$\ker \Phi = \langle \sum_{\varphi(\lambda)=u} a_\lambda t^\lambda, \sum a_\lambda = 0 \rangle$

$\subseteq$

$\langle t^a - t^b \mid \varphi(a) = \varphi(b) \rangle$

eqns of toric's are $x^{\lambda_1} = x^{\lambda_2}$

e.g. E ell. not toric, C_g, $g \geq 1$ not toric

$Gr(k,n)$ not toric,