Pts counting problem and RR.

Q: Given Q : polytope w/ $Q^{int} \neq \emptyset$

$\cap\!\!|$

$M_{\mathbb{R}}$

Find $\# \{kQ \cap M\}$, $k \in \mathbb{Z}_+$ (integral pts in kQ)

A: $\overset{!!}{?}$

Thm (Ehrhart): $f_Q(k)$ is a polynomial.

1st strategy: by Ehrhart, via pure combinatorics.

2nd strategy: by toric geometry.

Try: L ample l.b.

$\downarrow$

X proj. toric dim n

by Hilbert: $h^0(X, L^{\otimes k})$ is a polynomial

$\overset{/\!/}{H_L(z)|_{z=k}}$ (Hilb. poly) for $k>>0$.

leading coeff. $\frac{\deg L}{n!} z^n$

$L \rightsquigarrow \varphi_L : X \hookrightarrow \mathbb{P}^N$

$\mathcal{O}(L^{\otimes k}) = \mathcal{O}_{\mathbb{P}^N}(k)|_X =: \mathcal{O}_X(k)$

$R(X)$: homog. coord ring of X.

$\rightsquigarrow R(X)_k \longrightarrow H^0(X, \mathcal{O}_X(k))$

is an iso if $k>>0$.

If $X = X_Q \longleftrightarrow Q \subseteq M_{\mathbb{R}}$

(pol. by L)

$H^0(X, L^{\otimes k})$ is gen by integral pts in kQ

So $h^0(X, L^{\otimes k}) = \# \{kQ \cap M\}$

$\rightsquigarrow k \longmapsto \# \{kQ \cap M\}$

is a polynomial, $k>>0$.

Problem: how if k is small?

L l.b. $\quad \mathcal{F}$ coh. on X

$\downarrow$

X complete.

$\rightsquigarrow \chi(\mathcal{F} \otimes L^{\otimes k}) = \sum_{i=0}^{n} (-1)^i h^i(X, \mathcal{F} \otimes L^{\otimes k})$

Thm (Kleiman) $\exists$ poly. $H_{\mathcal{F},L}$, $\deg H_{\mathcal{F},L} \le n$

s.t. $\chi(\mathcal{F}\otimes L^{\otimes k}) = H_{\mathcal{F},L}(k)$

for $\underline{\text{all}}$ $k\in\mathbb{Z}$.

Choose $\mathcal{F} = \mathcal{O}_X$, L ample.

$$\chi(L^{\otimes k}) = H_L(k)$$

$$\| $$

$$h^0(L^{\otimes k}) - \underbrace{h^1(L^{\otimes k}) + h^2 \dots}_{\substack{\| \\ 0}}$$

vanishing thm on toric

e.g.

$Q = [0,1]^{\cancel{2}\, n}$

$\#\{kQ \cap M\} = (k+1)^{\cancel{2}\, n}$

Hirzebruch-Riemann-Roch

X : sm. complete dim n

$\mathcal{F}$: coh. on X

$$\chi(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i h^i(X,\mathcal{F})$$

Thm (HRR):

$$\chi(\mathcal{F}) = \int_X ch(\mathcal{F}) \cdot Td(X)$$

What is this?

- $\bullet$ = "$\cup$" cup product
- $\int_X$: take the top degree.

Chern class $c_i(\mathcal{F})$, $\mathcal{F}$ loc. free.

$$\cap$$

$$H^{2n-2i}(X,\mathbb{Z})$$

$$c(\mathcal{F}) = \underset{\substack{\| \\ 1}}{c_0(\mathcal{F})} + c_1(\mathcal{F}) + \dots + c_n(\mathcal{F})$$

total Chern

Usually just write c_i for $c_i(T_X)$

Important formula:

$$\chi(X) = \int_X c_n$$

topological Euler char.

What can we say if $X = X_\Sigma$, Σ complete?

Prop. 1). $c(T_X) = \prod_{\rho \in \Sigma(1)} (1 + D_\rho)$

$= \sum_{\text{all } \sigma} V(\sigma)$

2). $c_1(T_X) = c_1(\Lambda^n T_X) = c_1(\omega_X^\vee)$

$= -c_1(\omega_X) = -K_X$

$= \sum_{\rho \in \Sigma(1)} D_\rho$

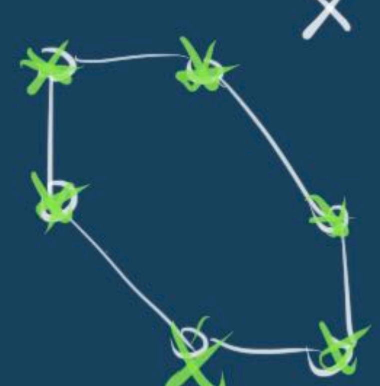
3). $c_n = c_n(T_X) = \#\{\sigma, \sigma \in \Sigma(n)\} \cdot pt$

Cor. $\int_X c_n = \int_X |\Sigma(n)| \cdot pt = |\Sigma(n)|$

since $\int_X pt = 1$

$\Rightarrow \chi(X) = |\Sigma(n)|$

e.g. $\left(\underset{X}{\underset{\shortparallel}{Bl_3\mathbb{P}^2}}, \ \pi^*\mathcal{O}(3) \otimes \left(\bigotimes_{i=1}^{3} \mathcal{O}(-E_i)\right)\right)$



$\chi(X) = 6$

Def: L is a l.b. on X, then its Chern character is

$H^\bullet(X, \mathbb{Q}) \ni ch(L) = 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \cdots$

$ch(L) = e^{c_1(L)}$

Todd class: we write

$c(T_X) = \prod_{i=1}^{n} (1 + \xi_i)$, w/ $\xi_i \in H^2(X, \mathbb{Q})$ called Chern roots.

$\Rightarrow$ c_i is the i-th elementary sym. polynomial of $\{\xi_1, \dots, \xi_n\}$

Def: $Td(X) = \prod_{i=1}^{n} \frac{\xi_i}{1 - e^{-\xi_i}}$ (= a sym. poly. in ξ_i's)

= polynomial in c_i's, $i = 1, \dots, n$

More explicitly:

$$\frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^k \frac{B_k}{(2k)!} x^{2k}$$

Bernoulli #'s

$$= 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots$$

e.g. X is a surface

$c_1 = \xi_1 + \xi_2$, $c_2 = \xi_1 \xi_2$

$$Td(X) = \prod_{i=1}^{2} \left(1 + \frac{1}{2}\xi_i + \frac{1}{12}\xi_i^2\right)$$

$$= 1 + \frac{1}{2}(\xi_1+\xi_2) + \frac{1}{12}(\xi_1^2+\xi_2^2+3\xi_1\xi_2)$$

$$= 1 + \frac{1}{2}(\xi_1+\xi_2) + \frac{1}{12}\left[(\xi_1+\xi_2)^2 + \xi_1\xi_2\right]$$

$$= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2+c_2)$$

c_i : deg i $\Rightarrow$ $Td(X) = \sum T_i$

T_i : homogeneous deg i

e.g. $T_0 = 1$, $T_1 = \frac{1}{2}c_1$, $T_2 = \frac{1}{12}(c_1^2+c_2)$

$T_3 = \frac{1}{24}c_1c_2$, ...

Now: $X = X_\Sigma$ complete ~~toric~~.

note: cohom. ring $H^\cdot(X, \mathbb{Z})$ is gen by D_ρ's, $\rho \in \Sigma(1)$.

Thm. $$Td(X) = \prod_{\rho \in \Sigma(1)} \frac{D_\rho}{1 - e^{-D_\rho}}$$

Pf: $c(T_X) = \prod_{\rho \in \Sigma(1)} (1 + D_\rho)$, $\underline{\Sigma(1) = \rho_1, \dots, \rho_r}$

elementary sym. poly

$\Rightarrow c_i = \sigma_i(D_1, \dots, D_r)$

Also. $c_i = \sigma_i(\xi_1, \dots, \xi_n)$

now if $r > n$?

Formally compute:

In $\mathbb{Q}[[x_1, \dots x_r]]$, $m = \langle x_1, \dots, x_r \rangle$

$$\prod_{i=1}^{r} \frac{x_i}{1-e^{-x_i}} = P(\sigma_1(x_1, \dots, x_r), \dots, \sigma_n(x_1, \dots, x_r))$$

for some P mod m^{n+1}

$$\Rightarrow \prod_{i=1}^{r} \frac{D_i}{1-e^{-D_i}} = P(c_1, \dots, c_n)$$

In $\mathbb{Q}[[x_1, \dots, x_n]]$, $m' = \langle x_1, \dots, x_n \rangle$

$$\frac{x_i}{1-e^{-x_i}} = 1 + \frac{1}{2}x_i + \cdots = 1 \text{ if } x_i = 0.$$

Let $x_i = 0$, $i = n+1, \cdots, r$.

$$\Rightarrow \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} = P(b_1(x_1, \cdots, x_r, 0, \cdots, 0), \cdots, b_n(x_1, \cdots, x_r, 0, \cdots, 0)) \mod m^{n+1}$$

$$= P(b_1(x_1, \cdots, x_r), \cdots, b_n(x_1, \cdots, x_r)).$$

Take $x_i = \xi_i$

$$\Rightarrow \prod_{i=1}^{n} \frac{\xi_i}{1-e^{-\xi_i}} = P(c_1, \cdots, c_n).$$

e.g. $X = X_{\mathbb{Z}}$ complete, dim 2.

20 min ago: $Td(X) = 1 + \frac{1}{2} c_1 + \frac{1}{12}(c_1^2 + c_2)$

HRR: D: div on X

$$\chi(\mathcal{O}_X(D)) = \int_X ch(\mathcal{O}_X(D)) \cdot Td(X)$$

Say $D = 0$, then $ch(\mathcal{O}_X) = 1$

$$\chi(\mathcal{O}_X) = \int_X (1 + \tfrac{1}{2} c_1 + \tfrac{1}{12}(c_1^2 + c_2)) = \int_X \tfrac{1}{12}(c_1^2 + c_2)$$

$$\int_X c_1^2 = (-K_X)\cdot(-K_X) = K_X^2$$

$$\int_X c_n = \chi(X)$$

$$\Rightarrow \chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + \chi(X))$$

(Noether's formula)

Say $D \neq 0$. $L = \mathcal{O}_X(D)$

$$ch(L) = 1 + c_1(L) + \frac{1}{2} c_1(L)^2$$

HRR:

$$\underline{\chi(\mathcal{O}_X(D))} = \int_X (1 + c_1(L) + \tfrac{1}{2}(c_1(L))^2) \cdot (1 + \tfrac{1}{2} c_1 + \tfrac{1}{12}(c_1^2 + c_2))$$

$$= \int_X \frac{1}{12}(c_1^2 + c_2) + \frac{1}{2} c_1(L) \cdot c_1 + \frac{1}{2} c_1(L)^2$$

$$= \chi(\mathcal{O}_X) - \frac{1}{2} D \cdot K_X + \frac{1}{2} D \cdot D$$

$$= \underline{\frac{1}{2}(D^2 - D \cdot K_X) + \chi(\mathcal{O}_X)}$$

is the classical RR for surfaces

Polytope Euler-Maclaurin formula (EM)

Classical: f : C^∞ on $[0,n]$

$$\Rightarrow \int_0^n f(x)\,dx = \frac{1}{2}f(0) + f(1) + \cdots + f(n-1) + \frac{1}{2}f(n) + \sum_{k=1}^{\cancel{l}\ \infty} (-1)^k \frac{B_k}{(2k)!}\left(f^{(2k-1)}\Big|_0^n\right) \cancel{+R}$$

if $|f^{(l)}(x)| < C\cdot\lambda^l$ for some $C>0$, $\lambda<2\pi$, $x\in[0,n]$, all $l\geq 1$.

Def: $$\mathrm{Todd}(x) = \frac{\frac{\partial}{\partial x}}{1-e^{-\frac{\partial}{\partial x}}} = 1 + \frac{1}{2}\frac{\partial}{\partial x} + \sum_{k=1}^{\infty}(-1)^k \frac{B_k}{(2k)!}\frac{\partial^{2k}}{\partial x^{2k}}$$

$$\left(\Rightarrow \mathrm{Todd}(x)(e^{xz}) \overset{(*)}{=} e^{xz}\cdot\frac{z}{1-e^{-z}}\right)$$

P : sm. polytope

$$P = \bigcap_{F} \{m \mid \langle m, u_F\rangle + a_F \geq 0\}$$

F: facet ; half space.

$h = (h_1, \cdots, h_d)$, $d = \#$ facets of P

$h_i \in \mathbb{R}$, $h_i \leftrightarrow F_i$

$\rightsquigarrow$ new polytope

$$P(h) = \bigcap_{F} \{m \mid \langle m, u_F\rangle + a_F + h_F \geq 0\}$$

if h_i's are small enough, then the normal of P, $P(h)$ are the same.

Denote $$\mathrm{Todd}(h) = \prod_F \frac{\frac{\partial}{\partial h_F}}{1-e^{-\frac{\partial}{\partial_F}}}$$

Thm (Khovanskii-Pukhlikov)

$$\mathrm{Todd}(h)\left(\int_{P(h)} e^{\langle x,z\rangle}\,dx\right)\Big|_{h=0} = \sum_{m\in P\cap M} e^{\langle m,z\rangle}$$

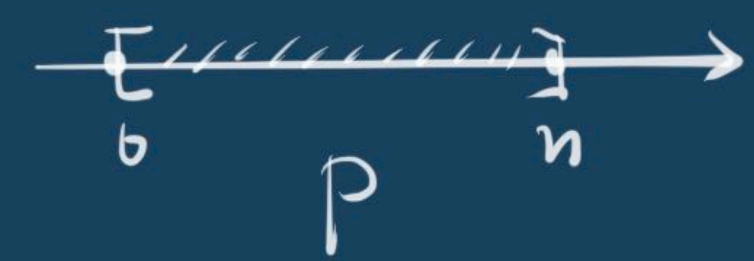
under some conditions $(*)$

KP Thm is polytope version of EM:

e.g. $P = [0, n] \subseteq \mathbb{R}$.

$\uparrow \quad \uparrow$

facets



normal vector: $\quad \overrightarrow{[} \qquad \overleftarrow{]}$

± 1.

P is given by $\langle m, 1\rangle \geq 0 \iff m \geq 0$

$\langle m, -1\rangle + n \geq 0 \iff m \leq n$

$h = (h_0, h_1) \rightsquigarrow P(h) = [-h_0, n+h_1]$

For $z \in \mathbb{C}$ satisfying the conditions $(0 < |z| < 2\pi)$.

$f(x) = e^{xz}$

$$\Rightarrow \mathrm{Todd}(h)\left(\int_{-h_0}^{n+h_1} f(x)\,dx\right)\Big|_{h_0=h_1=0} \quad (*)$$

$$\overset{\text{comp.}}{=} \int_0^n f(x)\,dx + \frac{1}{2}\left(f(0)+f(n)\right) \quad (1)$$

$$+ \sum \cdots$$

by KP

$$(*) = \sum_{m \in [0,n]\cap\mathbb{Z}} e^{mz} = f(0) + \cdots + f(n)$$

(2).

$(1)\ (2)$:

$$\int_0^n f(x)\,dx = \frac{1}{2}f(0) + f(1) + \cdots + f(n-1)$$

$$\frac{1}{2}f(n) + \sum \cdots$$

Cor: (of KM)

$$\mathrm{Todd}(h)\left(\mathrm{vol}(P(h))\right)\Big|_{h=0}$$

$$\shortparallel$$

$$|P \cap M|, \quad \mathrm{vol} = \frac{\mathrm{Vol}}{n!}$$

$\uparrow$

Euc. vol.

TRP $\Rightarrow$ THRR.

Sketch: $D = \sum a_\rho D_\rho \rightsquigarrow P_D =: P$
ample $\rightsquigarrow L = \mathcal{O}_X(D)$

$\deg(D) = \mathrm{Vol}(P)$
$\| $
$\int_X D^n$

$h = (h_1, \dots, h_{|\Sigma(1)|})$ is an $\mathbb{R}$-vector.

$\rightsquigarrow D(h) := \sum_{\rho \in \Sigma(1)} (a_\rho + h_\rho) D_\rho$

$\rightsquigarrow P_{D(h)} := \bigcap_{\rho \in \Sigma(1)} \{ m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle + a_\rho + h_\rho \geq 0 \}$
$\|$

- $P(h)$ is not lattice.
- h small, $P(h)$, P : same normal fan.
- $\int_X D(h)^n = \mathrm{Vol}(P(h))$ ---- $\star\star$

$$\mathrm{Todd}(h_\rho)(e^{h_\rho D_\rho}) = \frac{D_\rho e^{h_\rho D_\rho}}{1 - e^{-D_\rho}} \quad \star$$

by $\circledast$

$\mathrm{Todd}(h)(e^{D(h)})\Big|_{h=0}$

$= \prod_\rho \mathrm{Todd}(h_\rho)\, e^{(a_\rho + h_\rho) D_\rho}\Big|_{h=0}$

$\overset{\star}{=} \prod_\rho \frac{D_\rho e^{a_\rho D_\rho}}{1 - e^{-D_\rho}}$

$= \mathrm{ch}(L)\,\mathrm{Td}(X)$

Recall: $\mathrm{ch}(L) = e^{\sum a_\rho D_\rho}$

$\mathrm{Td}(X) = \prod \frac{D_\rho}{1 - e^{-D_\rho}}$

Take top degree:

$\int_X \mathrm{ch}(L)\,\mathrm{Td}(X)$

$\overset{\circledast}{=} \int_X \mathrm{Todd}(h)\, e^{D(h)}\Big|_{h=0}$

$= \mathrm{Todd}(h)\left(\int_X e^{D(h)}\right)\Big|_{h=0}$

$= \mathrm{Todd}(h)\left(\int_X \frac{D(h)^n}{n!}\right)\Big|_{h=0}$

$\overset{\star\star}{=} \mathrm{Todd}(h) \cdot \frac{1}{n!} \mathrm{Vol}(P(h))\Big|_{h=0}$

Cor
$= |P \cap M|$

Ehrhart
$= \chi(L) = \chi(\mathcal{O}_X(D))$

THRR.