Cohomology (on toric var's)

X: var $\mathcal{F}$: sheaf on X

$\rightsquigarrow$ injective resolution

$$0 \to \mathcal{F} \to I_0 \to I_1 \to \cdots$$

I_i's are inj. $\forall$

$$\begin{tikzcd} 0 \arrow[d] & \\ G \arrow[r] \arrow[d] & I_i \\ H \arrow[ur, dashed] & \end{tikzcd}$$

Apply $\Gamma(X, \cdot)$:

$$0 \to \Gamma(X,\mathcal{F}) \to \Gamma(X, I_0) \xrightarrow{d^0} \Gamma(X, I_1) \xrightarrow{d^1} \cdots$$

$$H^i(X,\mathcal{F}) = \ker d^i / \operatorname{im} d^{i-1}$$

adv: general enough

disadv: not easy for computation

Instead, people prefer Čech cohom.

X is covered by $\mathcal{U} = \{U_i\}_{i=1}^{l}$, U_i open $\subseteq X$

notation: $[l]_p :=$ all $(p+1)$-tuples of $\{1, \ldots, l\}$

p-th Čech cochain: $$\check{C}^p(\mathcal{U},\mathcal{F}) := \bigoplus_{(i_0,\ldots,i_p)\in[l]_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p})$$

$$d^p : \check{C}^p(\mathcal{U},\mathcal{F}) \to \check{C}^{p+1}(\mathcal{U},\mathcal{F})$$

$$d^p(\alpha)_{i_0\ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0\ldots\hat{i_k}\ldots i_{p+1}}\Big|_{i_0\ldots i_{p+1}}$$

$\rightsquigarrow$ $\check{H}^p(\mathcal{U},\mathcal{F}) := \ker d^p / \operatorname{im} d^{p-1}$

If U_i's are all affine, then

$\Downarrow$ Serre vanishing

$p>0$, $H^p(\text{affine}, \text{q-coh}) = 0$

Thm. $\check{H}^p(\mathcal{U},\mathcal{F}) \cong H(X,\mathcal{F})$ where $\mathcal{F}$ is q-coh.

e.g. $X = \mathbb{P}^1$, $\mathcal{F} = \Omega^1_{\mathbb{P}^1}$

by Euler sequence:

$$0 \to \Omega^1_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(-1)^2 \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

$$0 \to \underset{=0}{H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1})} \to \underset{=0\,?}{\bigoplus_2 H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))} \to \cdots$$

$$\to H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \to \underset{=0\,?}{\bigoplus_2 H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))}$$

$$\Rightarrow H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \cong \underset{=\mathbb{C}}{H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})}$$

$\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}, \quad \mathcal{O}_{\mathbb{P}^1}(-1)$

$$\begin{array}{ccccc} X & = & \mathcal{U}_0 & \cup & \mathcal{U}_1 \\ \| & & \| & & \| \\ \mathbb{P}^1 & & \operatorname{Spec}\mathbb{C}[x] & & \operatorname{Spec}\mathbb{C}[\frac{1}{x}] \end{array}$$

$$\mathbb{C}^* = \mathcal{U}_0 \cap \mathcal{U}_1 = \operatorname{Spec}\mathbb{C}[x, \tfrac{1}{x}]$$

Čech cochain : $\mathcal{F}(\mathcal{U}_0) \oplus \mathcal{F}(\mathcal{U}_1) \xrightarrow{d^0} \mathcal{F}(\mathcal{U}_0 \cap \mathcal{U}_1)$

$H^p(\mathbb{P}^1, \mathcal{F}) = 0$ if $p \geq 2$.

• For $\mathcal{O}_{\mathbb{P}^1}$:

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^1}(\mathcal{U}_0) & \oplus & \mathcal{O}_{\mathbb{P}^1}(\mathcal{U}_1) & \xrightarrow{d^0} & \mathcal{O}_{\mathbb{P}^1}(\mathcal{U}_0 \cap \mathcal{U}_1) \\ \| & & \| & & \| \\ \mathbb{C}[x] & & \mathbb{C}[\frac{1}{x}] & & \mathbb{C}[x, \frac{1}{x}] \end{array}$$

$$(f(x), \ g(\tfrac{1}{x})) \mapsto f(x) - g(\tfrac{1}{x})$$

If $d^0(f, g) = 0 \Rightarrow f(x) = g(\frac{1}{x})$

$\Rightarrow f = g = c \in \mathbb{C}$

$\Rightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$

d^0 is surj $\Rightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$.

$$k[x, \tfrac{1}{x}] \ni h = \sum a_i x^i = \underbrace{\sum_{i<0} a_i x^i}_{= -g(\frac{1}{x})} + \underbrace{\sum_{i\geq 0} a_i x^i}_{= f(x)}$$

• For $\mathcal{O}_{\mathbb{P}^1}(-1)$, consider

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_D \to 0$$

D is $[1:0]$

$\Rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(-1)) = H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$

In toric world, goal:

given $X = X_\Sigma$

D: toric div. on X

$\mathcal{F} = \mathcal{O}_X(D)$

compute $H^p(X, \mathcal{F}) = ?$

$D = \sum a_\rho D_\rho, \quad \rho \in \Sigma(1)$

$\exists$ natural affine open cover of X:

$\mathcal{U} = \{\mathcal{U}_\sigma\}_{\sigma \in \Sigma(n)}$, need σ's ordered.

(cone $\sigma \leftrightarrow$ affine open)

$$\check{C}(\mathcal{U}, \mathcal{O}_X(D)) = \bigoplus_{(i_0 \cdots i_p) \in [l]_p} H^0\Big(\bigcap_{k=0}^{p} \mathcal{U}_{i_k}, \mathcal{O}_X(D)\Big)$$

Recall: $D = \sum a_\rho D_\rho \rightsquigarrow$ polytope / polyhedron $P(D)$

$$M_{\mathbb{R}} \supseteq P(D) := \{ m \in M_{\mathbb{R}} \mid \langle m, v_\rho \rangle \geq -a_\rho, \ \forall \rho \in \Sigma(1) \}.$$

v_ρ's are primitive vectors in ρ.

then $$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{m \in P(D) \cap M} \mathbb{C}\chi^m.$$

So $$H^0(U_\sigma, \mathcal{O}_X(D)) = \bigoplus_m H^0(U_\sigma, \mathcal{O}_X(D))_m$$

where $$H^0(U_\sigma, \mathcal{O}_X(D))_m = \begin{cases} \mathbb{C}\chi^m, & \text{if } \langle m, v_\rho \rangle \geq -a_\rho \ \forall \rho \in \Sigma(1) \quad (*) \\ 0, & \text{otherwise.} \end{cases}$$

is M-graded

$\Downarrow$

$\check{C}^p(\mathcal{U}, \mathcal{O}_X(D))$ is M-graded

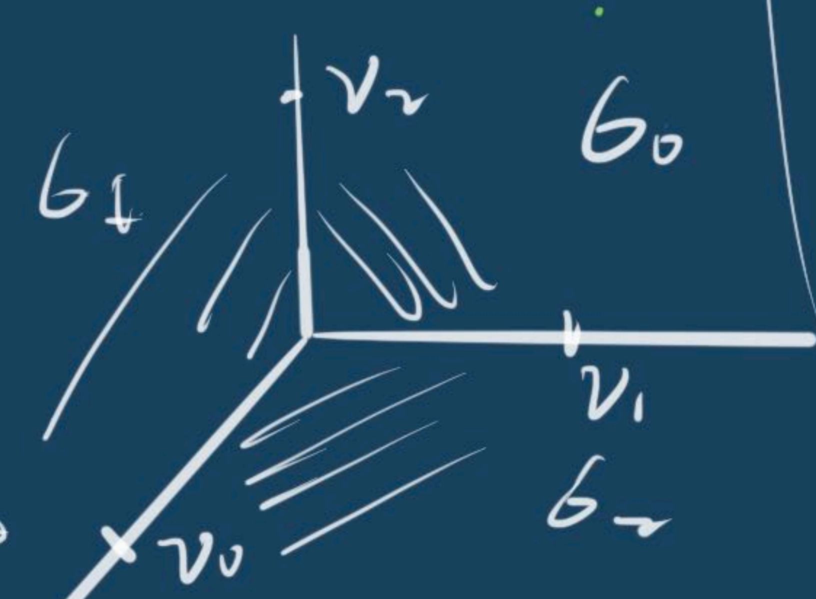
$\Downarrow$

$\check{H}^p(\mathcal{U}, \mathcal{O}_X(D))$ ... ...

$\Downarrow$

$H^p(X, \mathcal{O}_X(D))$ ... ...

e.g. $X = \mathbb{P}^2$, Σ:



$D_0 \longleftrightarrow v_0$

Compute $H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$

$dH = dD_0$

$\sigma_i \longleftrightarrow U_i \ (\cong \mathbb{A}^2)$

U_{ij}

$U_{012} \ (\cong (\mathbb{C}^*)^2)$

Čech complex: (by d I mean $\mathcal{O}_{\mathbb{P}^2}(d)$)

$$0 \to \check{C}^0(d) \xrightarrow{d^0} \check{C}^1(d) \xrightarrow{d^1} \check{C}^2(d) \to 0$$

$$\check{C}^0(d) = \bigoplus_{i=0}^{2} H^0(U_i, \mathcal{O}(d))$$

$$\check{C}^1(d) = \bigoplus_{i<j} H^0(U_{ij}, \mathcal{O}(d))$$

$$\check{C}^2(d) = H^0(U_{012}, \mathcal{O}(d))$$

$$d^0 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \qquad d^1 = (1 \ -1 \ 1)$$

To see $(*)$

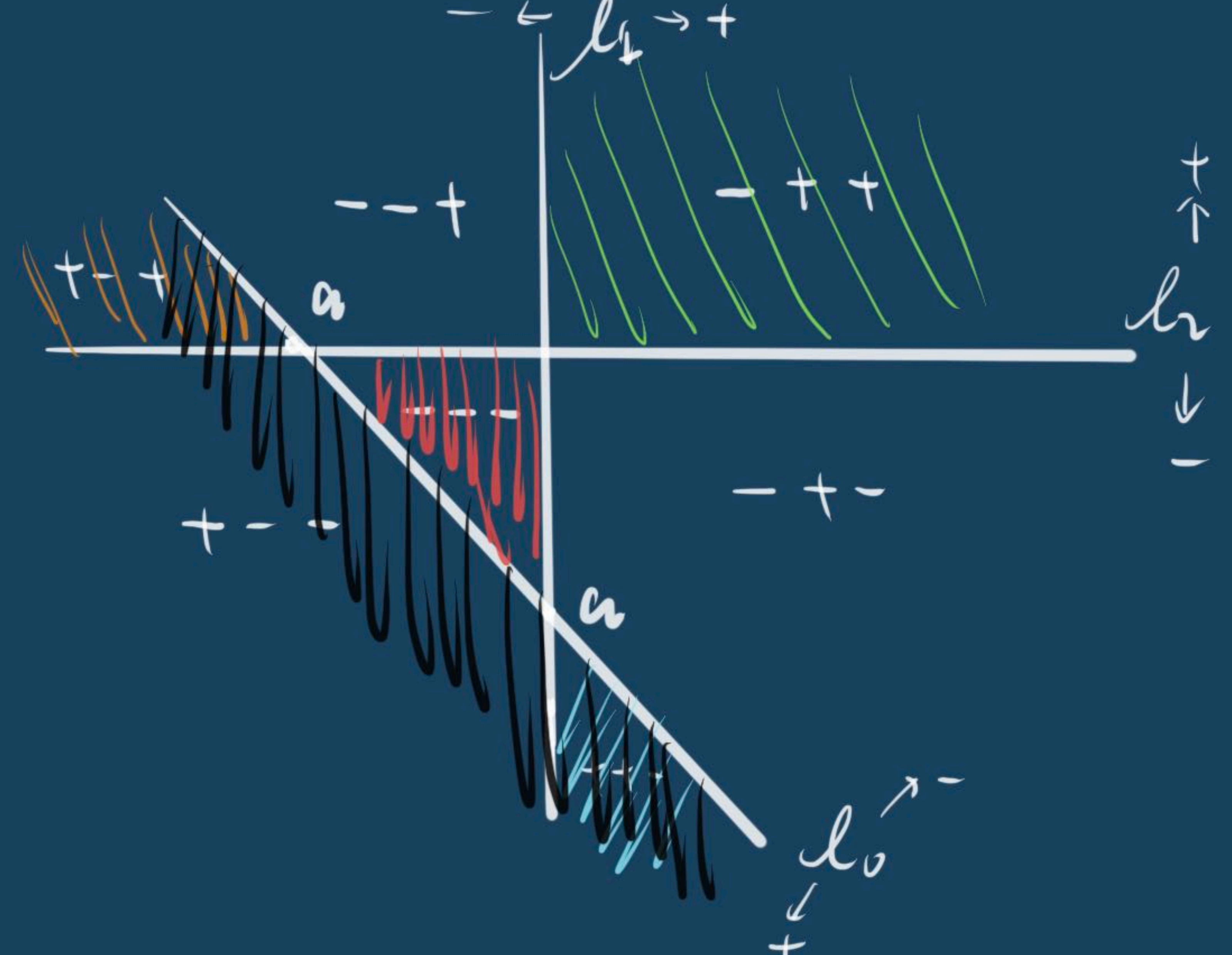
$l_0 = \{ m \in M_{\mathbb{R}} \mid \langle m, v_0 \rangle = -d \}$

$l_1 = \{ \ldots \mid \langle \ldots, v_1 \rangle = 0 \}$

$l_2 = \{ \ldots \mid \ldots \ v_2 = 0 \}$.

$l_{0,1,2}$ cut $M_{\mathbb{R}}$ into chambers.

Assume $d < 0$



notation: $C_{+--} \Leftrightarrow \langle m, v_0\rangle \ge -d$
$\langle m, v_1\rangle < 0$
$\langle m, v_2\rangle < 0$

e.g. $C_{-++} \Leftrightarrow$ $< -d$
≥ 0
≥ 0

• $\sigma_i = \langle v_j, v_k\rangle$ $\{i,j,k\} = \{0,1,2\}$.
cone gen. by

$H^0(U_0, \mathcal{O}(d))_m \neq 0 \Leftrightarrow m \in C_{-++} \cap M$
" U_1 " " " $\Leftrightarrow C_{+-+}$
U_2 $\Leftrightarrow C_{++-}$

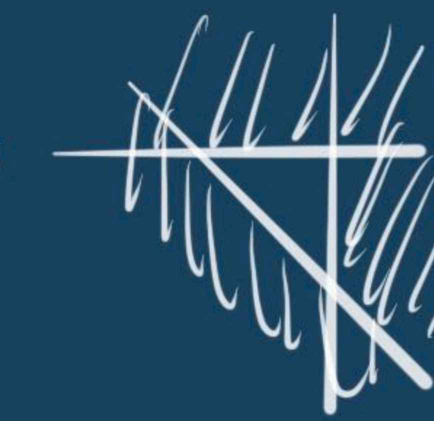
in these chambers $\check{C}^0(d)_m \neq 0$

• $\sigma_i \cap \sigma_j = \langle v_k \rangle$

$\underbrace{\sigma_1 \cap \sigma_2}_{\tau} = \langle v_0 \rangle$ dual to $\underbrace{\langle m, v_0\rangle \ge -d}_{\text{in } M_{\mathbb{R}}}$

σ_{12} similarly σ_{02}, σ_{01}

$\check{C}^1(d)_m \neq 0 \Leftrightarrow m$ in



• $\sigma_{012} \leftrightarrow U_{012} (\cong (\mathbb{C}^*)^2)$

$\check{C}^2(d)_m \neq 0$ always true.

In each m-grading:

m in	dim $\check{C}^0(d)$	" 1 "	" 2 "
$C_{-++} \cup C_{+-+} \cup C_{++-}$	1	2	1
$C_{-++} \cup C_{-+-} \cup C_{+--}$	0	1	1
C_{---}	0	0	1

problem
others are exact.

means $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))_m \neq 0$.

only $m \in$  contributes.
interior

$$H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = \bigoplus_{m\in Q^{int}} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))_m$$

$\Rightarrow h^2(\mathcal{O}_{\mathbb{P}^2}(d)) = \#\ \mathbb{Z}\text{-pt in } Q^{int}$ $\quad Q=$

$= \binom{-d-1}{2}$

similarly, if $d > 0$

$$h^p(\mathcal{O}_{\mathbb{P}^2}(d)) = \begin{cases} \binom{d+2}{2}, & p=0 \\ 0, & p>0 \end{cases}$$

Computation (in general).

$X = X_\Sigma$, $\quad D = \sum a_\rho D_\rho$, $\quad m \in M$

def: a) $V_{D,m} = \bigcup_{\sigma\in\Sigma} \mathrm{Conv}\{v_\rho \mid \rho \in \sigma(1), \langle m, v_\rho\rangle < -a_\rho\}$

b). if D Cartier then $\longleftrightarrow$ $\exists$ piecewisely linear φ_D on Σ

$V^{supp}_{D,m} = \{u \in |\Sigma| \mid \langle m, u\rangle < \varphi_D(u)\}$

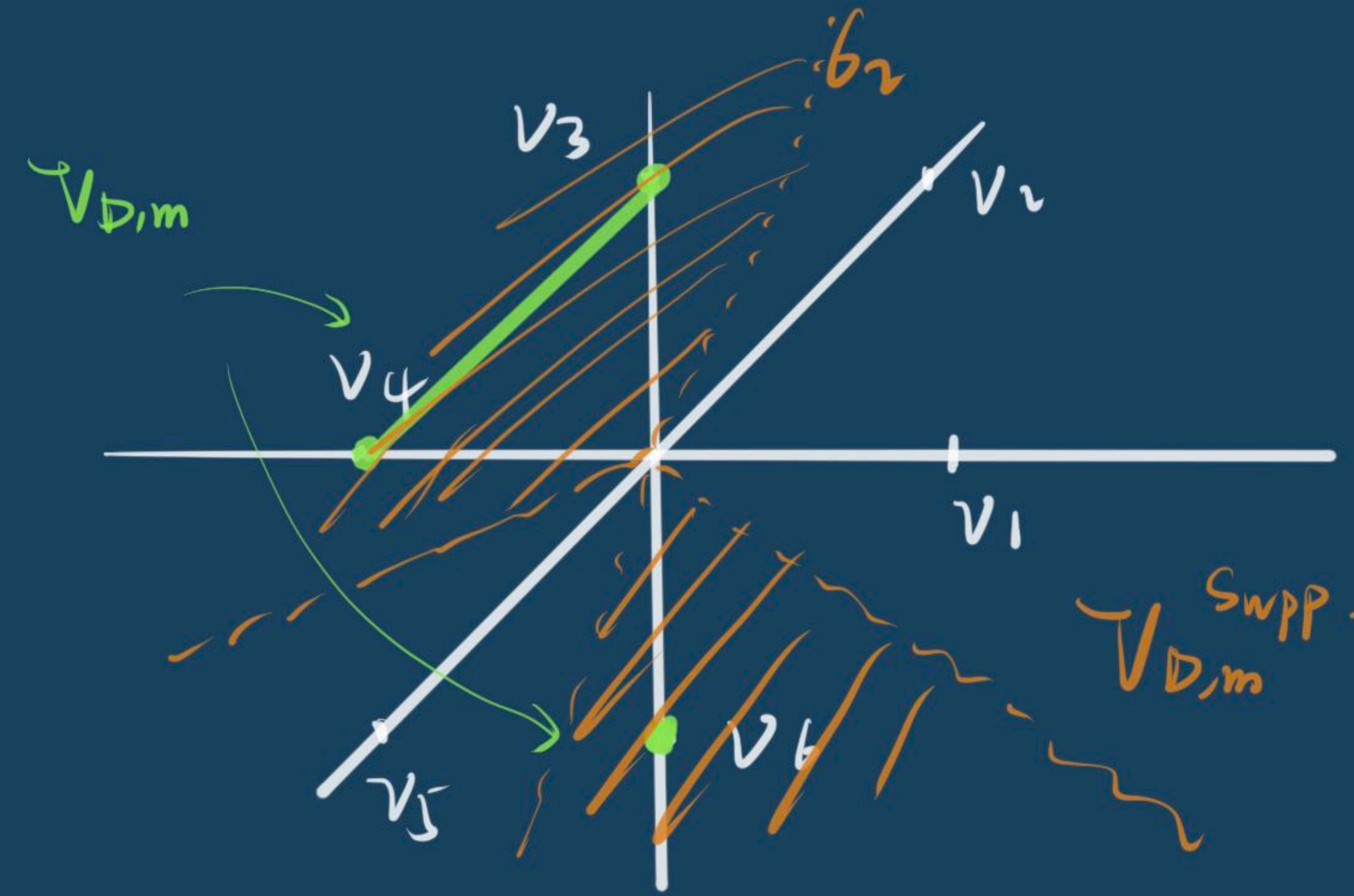
expect. $(1,0)(u_1,u_2) < \underset{\overset{\shortparallel}{m_2}}{(-1,1)}\cdot(u_1,u_2)$

$2u_1 < u_2$

e.g. $X_\Sigma = Bl_3\mathbb{P}^2$ $\quad P_D$:

$D = -D_3 + 2D_5 - D_6$

$m = e_1 = (1,0) \in M$



$\langle m, v_1\rangle = (1,0)\cdot(1,0) = 1 \geq 0$

$\langle m, v_2\rangle = (1,0)\cdot(1,1) = 1 \geq 0$

$\langle m, v_3\rangle = (1,0)\cdot(0,1) = 0 < -a_3 = 1$

$\langle m, v_4\rangle = (1,0)\cdot(-1,0) = -1 < -a_4 = 0$

$\langle m, v_5\rangle = (1,0)\cdot(-1,-1) = -1 \geq -a_5 = -2$

$\langle m, v_6\rangle = (1,0)(0,-1) = 0 < -a_6 = 1$

$\sigma_i = \langle v_i, v_{i+1}\rangle$ cyclic: $v_7 = v_1$

in σ_2 : m_2

$\langle m_2 \cdot v_2\rangle = 0$, $\quad \langle m_2\cdot v_3\rangle = 1$

$m_2 = (-1, 1)$

By using $V_{D,m}$, $V_{D,m}^{supp}$, we have:

Thm: D is only Weil

$\Rightarrow H^p(X, \mathcal{O}_X(D))_m \cong \tilde{H}^{p-1}(V_{D,m}, \mathbb{C})$

D is Cartier

$\Rightarrow H^p(X, \mathcal{O}_X(D))_m \cong \tilde{H}^{p-1}(V_{D,m}^{supp}, \mathbb{C})$

Remember: $\tilde{H}^p(Y, R) = \begin{cases} \tilde{H}^0(Y, R) = \operatorname{coker}\left(\begin{matrix} R \\ \| \\ H^0(pt, R) \\ \downarrow \\ H^0(Y, R) \end{matrix}\right), & p = 0. \\ H^p(Y, R), & p > 0 \end{cases}$

R: coeff ring.

$Y \neq \emptyset$

Main app. in dim 2.

X_Σ: smt. $D = \sum a_\rho D_\rho$ toric div.

$h^p(\mathcal{O}_X(D))$ computation $\xhookrightarrow{\text{reduce}}$ (counting problem)

how?

$l_\rho := \{m \in M_{\mathbb{R}} \mid \langle m, v_\rho \rangle = -a_\rho\}$

$\{l_1, \cdots, l_{|\Sigma(1)|}\}$ cut $M_{\mathbb{R}}$ into chambers, each chamber

$\downarrow$

$\operatorname{sign}_D(m)$

sth like $+-+--$.

(cyclic)

Prop. $\cdot\, h^0(\mathcal{O}_X(D))_m = \begin{cases} 1, & \operatorname{sign}_D(m) = +\cdots++ \\ 0, & \text{otherwise}. \end{cases}$

$\cdot\, h^1(\mathcal{O}_X(D))_m = \max\{0, \#\text{ conn. cpts of } V_{D,m} - 1\}$

$= \max\{0, \#\text{ strings of consecutive "$-$"s} - 1\}$

e.g. $--+-++-+-$ (conn.)

# of strings of cons. "$-$" is 3

$\cdot\, h^2(\mathcal{O}_X(D))_m = \begin{cases} 1, & \operatorname{sign}_D(m) = -\cdots- \\ 0, & \text{otherwise}. \end{cases}$

When people compute H^p's.

an important tool is Serre duality.

Thm. X complete dim n toric

D: $\mathbb{Q}$ - Cartier div

$\Rightarrow H^p(X, \mathcal{O}_X(D))^\vee \cong H^{n-p}(X, \mathcal{O}_X(K_X - D))$

Q: When are we safe?

A: X is CM.

Yes, normal toric var's are CM.

Actually, $\forall$ ample D on X

$\circledast$ X CM $\Leftrightarrow H^p(X, \mathcal{O}(-kD)) = 0$

$\forall p < n,\ k >> 0$

Thm (Batyrev - Borisov)

X_Σ complete toric, D: $\mathbb{Q}$- Cartier

D nef $\Rightarrow H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(-D)) = 0$, $\forall p \neq \dim P(D)$

D ample

$\Downarrow$

kD nef.

$\Downarrow$. BB

$H^p(X_\Sigma, \mathcal{O}_X(-kD)) = 0$

$\Downarrow$. $\circledast$

X CM.