Previously:

in each cone σ, primitive v's of $\sigma(1)$ on the same hyp.pl.

$\Updownarrow$

A): $f: X \to Y$ proj, toric, X is $\mathbb{Q}$-Gorenstein, X: dim n.

R: K_X-neg. extr. ray in $\overline{NE}(X/Y)$

$\downarrow$

$\varphi_R: X \to W$ contraction along R.

$\Rightarrow$ if φ_R is birational, then

$$\ell(R) < \max \dim \varphi_R^{-1}(w) + 1$$

$$\ell(R) = \min_{[C]\in R} -K_X \cdot C, \qquad \max \dim \varphi_R^{-1}(w) = d \ (\le n-1)$$

B): X complete toric, D: $\mathbb{Q}$-Cartier div.

D is ps.eff.

$\Updownarrow$

$\exists m \in \mathbb{Z}_{>0}$ s.t. $H^0(X, \mathcal{O}_X(mD)) \neq 0$

$\Updownarrow$

$(\kappa(X, D) \geq 0)$

Goal:

Thm. X dim n. proj. $\mathbb{Q}$-Goren. ~~toric~~.

D: $\mathbb{Q}$-div. ample Cartier

i) $\Rightarrow$ · $K_X + (n-1)D$ ps. eff

$\Updownarrow$

$K_X + (n-1)D$ nef. ✓

ii) $\Rightarrow$ · if X is Gorenstein

$H^0(X, \mathcal{O}_X(K_X + (n-1)D)) \neq 0$

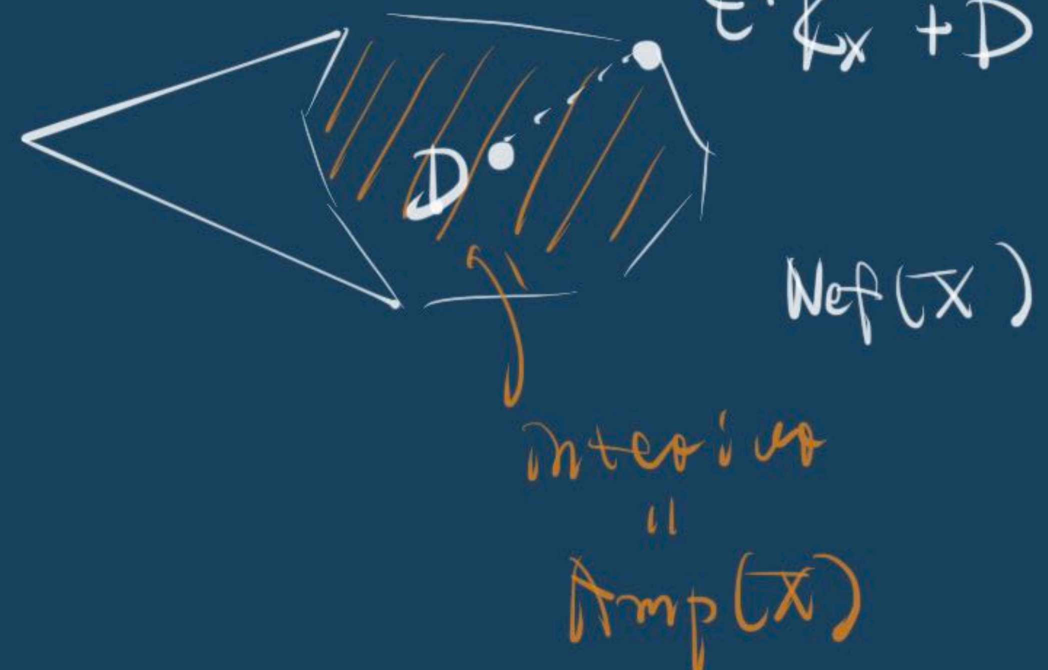
$\Downarrow$

the complete linear sys. $|K_X + (n-1)D|$ BPF.

i): pf: only "$\Downarrow$" is nontrivial.

$(\mathrm{Nef}(X) \subseteq \mathrm{PE}(X))$

nef threshold $:= \tau^{-1}$



take $\tau \in \mathbb{Q}$ s.t.

$K_X + \tau D$ is nef

but not ample

R: extr. ray choose rat.

$\cap$ C, $[C] \in R$.

$\overline{NE}(X/Y)$

$(K_X + \tau D) \cdot C = 0$

$\Downarrow$

$-K_X \cdot C = \tau D \cdot C > n-1$.

· If $\tau \le n-1$. $K_X + (n-1)D = \underbrace{K_X + \tau D}_{\text{nef}} + \underbrace{(n-1-\tau)D}_{\text{ample}}$

Done! nef $\Leftarrow$

· If $\tau > n-1$. Choose m s.t.

$m(K_X + \tau D)$ is Cartier.

$\varphi_{|m(K_X+\tau D)|} =: f : X \longrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(m(K_X+\tau D)))^\vee)$

w/ image Y.

By the assumption, $K_X + \tau D$ not ample.

$\Downarrow$

f is not an iso $X \to Y$.

Actually, it is worse.

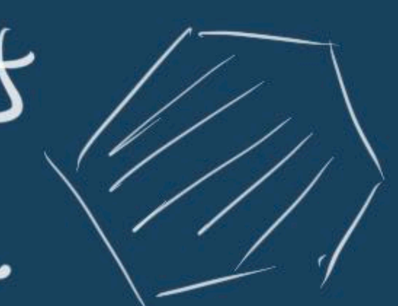
By the result A).

f is not birational.

i.e. $K_X + \tau D$ is not big.

But: $\overline{\mathrm{Big}(X)} = PE(X)$

$\Rightarrow K_X + \tau D$ is on the boundary of



$K_X + (n-1)D = (K_X + \tau D) -$

and $\underbrace{(\tau - (n-1))D}_{\text{ample}}$

$\notin PE(X)$

contradiction.

ii). $H^0(X, \mathcal{O}_X(K_X+(n-1)D))\neq 0$

$\Downarrow$ B).

$K_X+(n-1)D$ ps. eff

$\Downarrow$ i)

$K_X+(n-1)D$ nef.

$\Downarrow$?

$|K_X+(n-1)D|$ BPF.

For Cartier div's on toric var.

nef $\Leftrightarrow$ semiample.

BPF part. i.e. ii) is the toric version of Fujita conj.

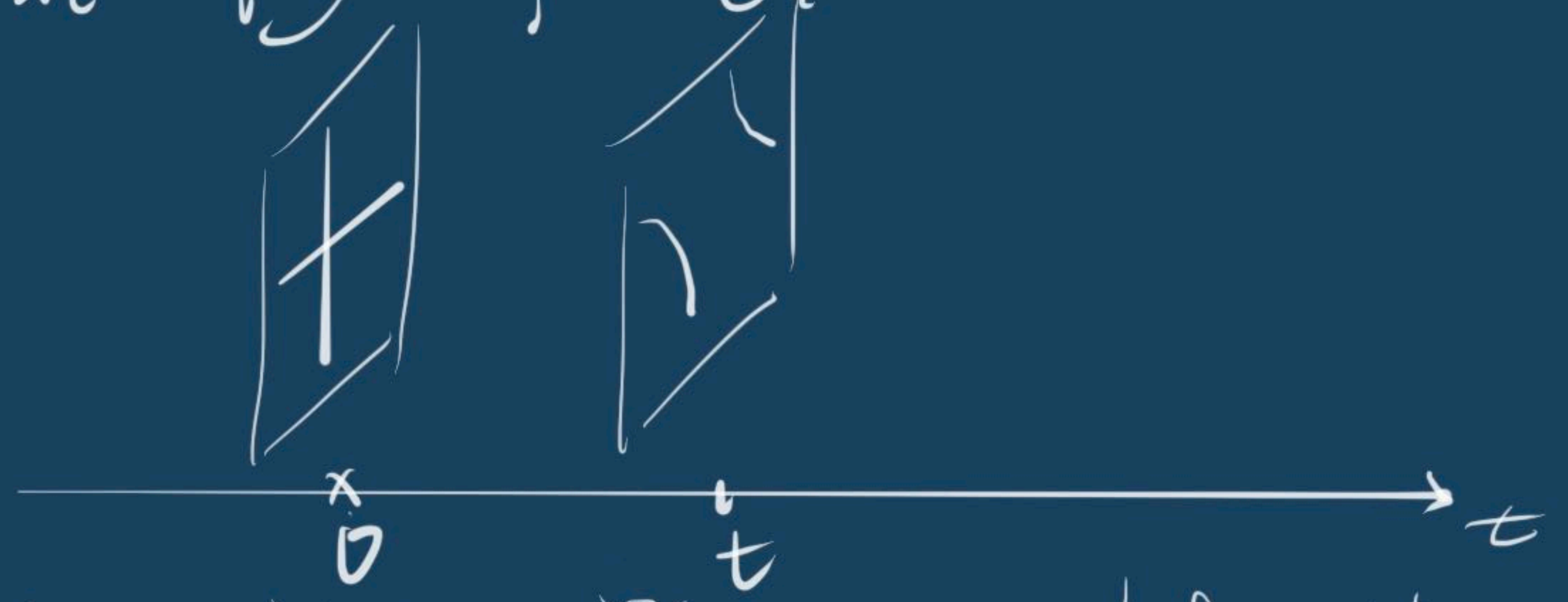
Deformation (in a naive way). of toric var's.

Warm up: $X_0=\{xy=0\}\subseteq\mathbb{C}^2$

deform X_0 how?

by perturbing the equation.

$X_t=\{xy=t\}\subseteq\mathbb{C}^2_t$



$t\to 0$, $X_t \rightsquigarrow X_0$

degenerate

deformation

$\updownarrow$ opposite

degeneration.

More general: $X_0=\{f(x)=0\}\subseteq\mathbb{C}^N$

hyp. surf.

perturb: $X_t=\{f(x)+t_1g_1(x)+\cdots+t_mg_m(x)=0\}\subseteq\mathbb{C}^N$

for each set of polynomials

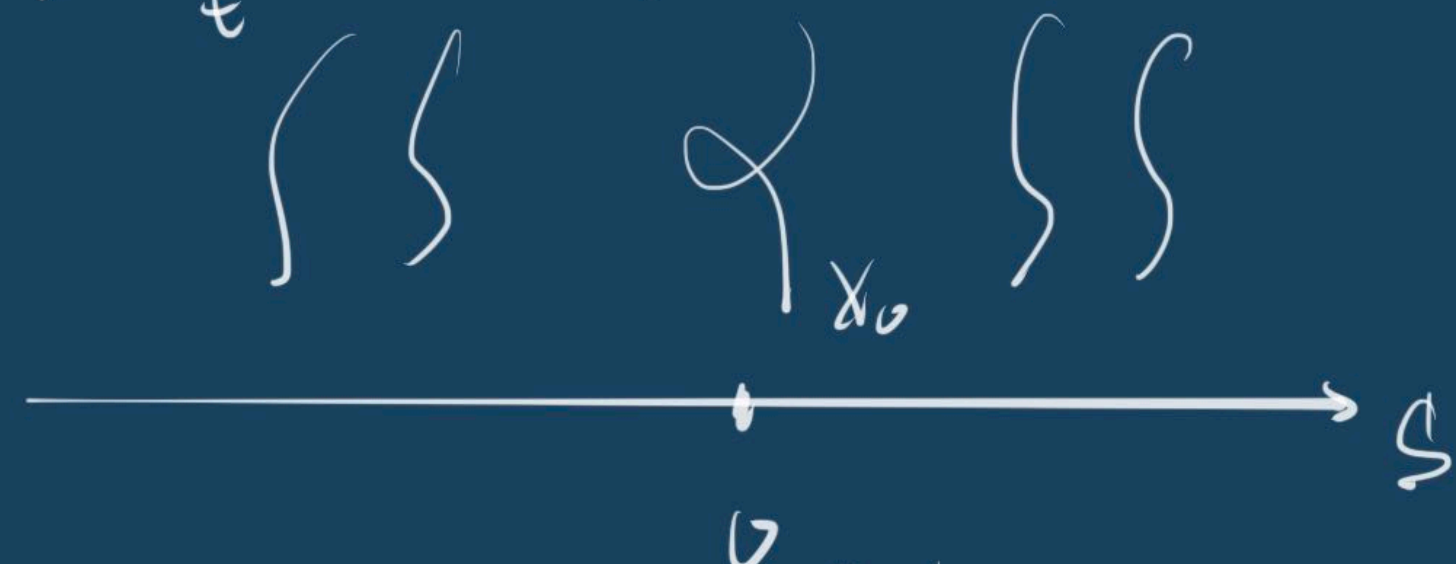
$g_1,\cdots,g_m$.

$t=(t_1,\cdots,t_m)\in\mathbb{C}^m$

Thus, we have

$$\begin{tikzcd} \{f+\sum t_i g_i = 0\} =: \mathcal{X} \arrow[d, "\pi"] & \subseteq \mathbb{C}^\nu_x \times \mathbb{C}^m_t \arrow[dl, "p_2"] \\ S := \mathbb{C}^m_t & \end{tikzcd}$$

$\mathcal{X} = \bigcup_t \mathcal{X}_t$, $\pi^{-1}(t) = \mathcal{X}_t$



X_0: special fiber.

Strictly:

Def: X: alg var.

A deformation of X is a map

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"] \\ S \end{tikzcd} \quad \text{s.t.}$$

. $\pi^{-1}(0) = X$

. π is flat.

Rmk: In case $S = \mathbb{C}^m$, then

flatness $\Leftrightarrow$ the special fiber X is a relative complete intersection in $\mathcal{X}$.

alg: $X = V(I)$

I is gen. by a regular sequence

$I = \langle f_1, \cdots, f_k \rangle$

$f_i \in H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$

Toric case:

X: affine, toric

Def: $f_1, \cdots, f_m \in H^0(X, \mathcal{O}_X)$ is called a toric reg. seq. if

i). f_i's are binomial in $H^0(X, \mathcal{O}_X)$

ii). $V(f_1, \cdots, f_m) \subseteq X$

is . affine toric

. codim m in X

($\Rightarrow$ $f_1, \cdots, f_m$ is a reg. seq. in X).

Example: $\mathbb{P}^1 \xrightarrow{v} \mathbb{P}^4$ $(\Leftrightarrow (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)))$.
Veronese $\Updownarrow$ polytope.

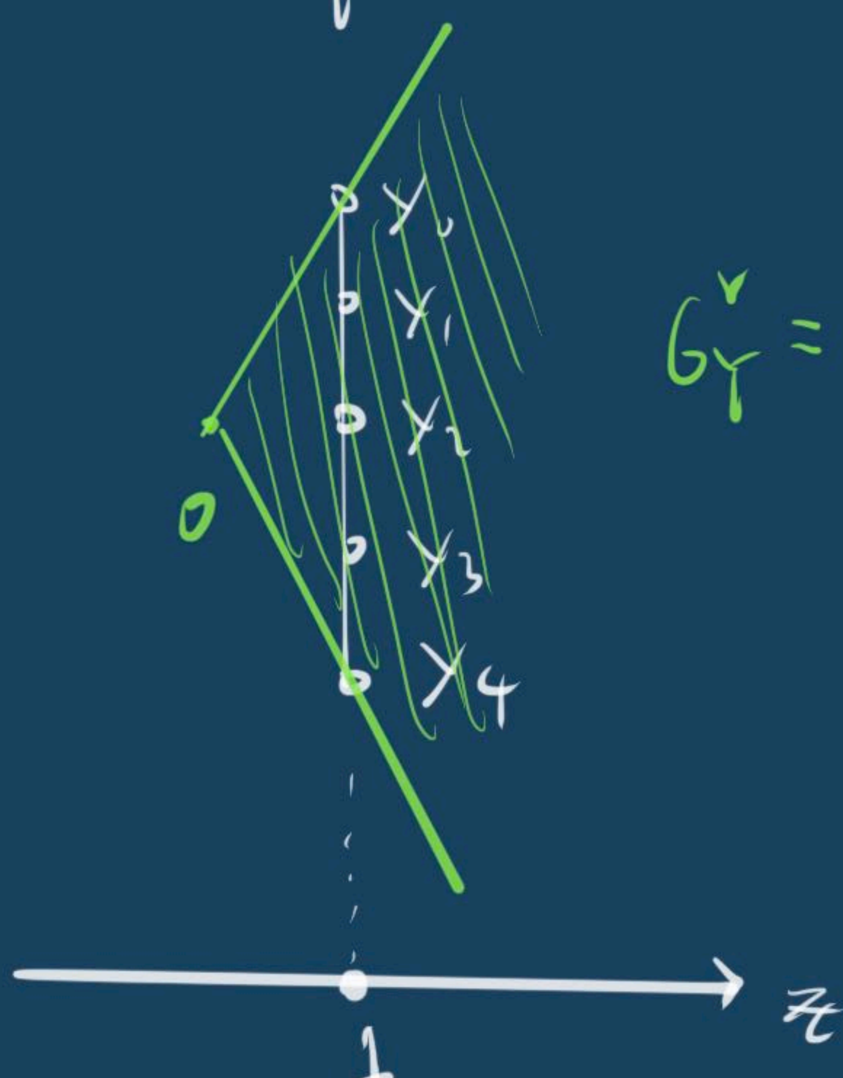
$[x_0 : x_1] \longmapsto [x_0^4 : x_0^3 x_1 : x_0^2 x_1^2 : x_0 x_1^3 : x_1^4]$

$ \underset{y_0}{\shortparallel} \quad \underset{y_1}{\shortparallel} \quad \underset{y_2}{\shortparallel} \quad \underset{y_3}{\shortparallel} \quad \underset{y_4}{\shortparallel}$

equvly: Y = affine cone over $v(\mathbb{P}^1)$ is given by

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \overset{\leq 1}{=} 1$$

Toric picture:

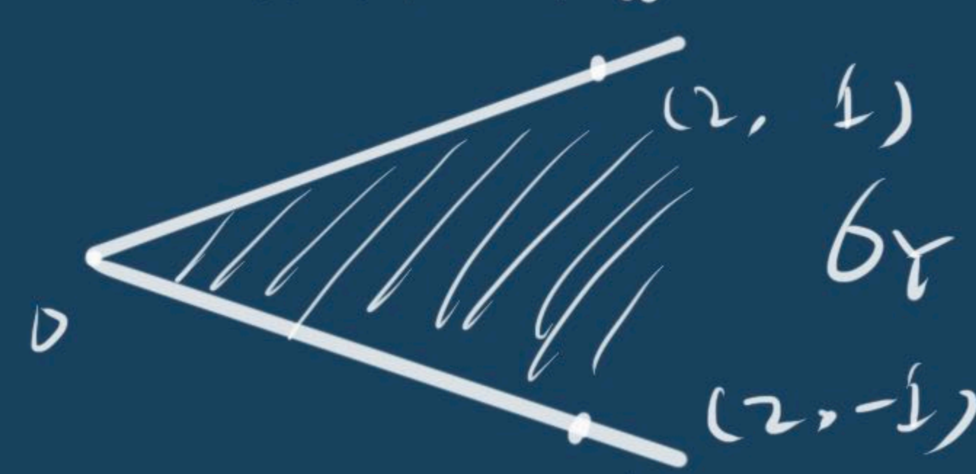


$M_{\mathbb{R}} \subseteq \widetilde{M}_{\mathbb{R}} = (\mathbb{Z} \oplus M) \otimes_{\mathbb{Z}} \mathbb{R}$.

Cone over

$\sigma_Y^\vee = \langle (1,-2), (1,2) \rangle$

on the dual side:



in $\widetilde{N}_{\mathbb{R}}$

Deform Y, by perturbing

$\mathcal{Y}$: affine cone over

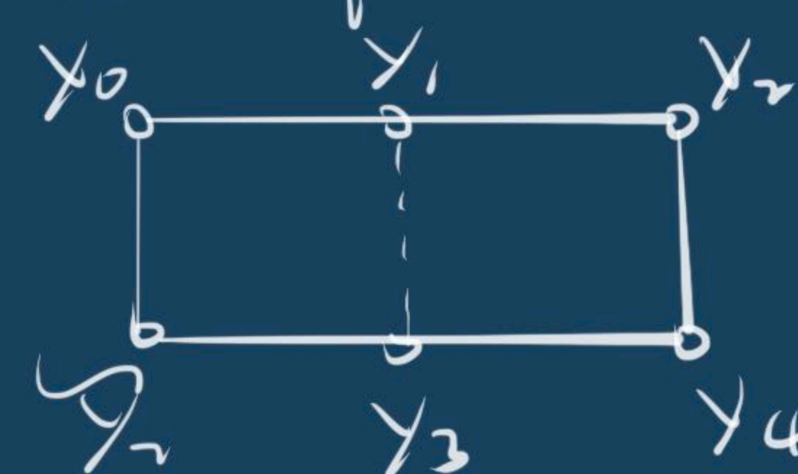
$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & \tilde{y}_2 = y_2 + t & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1$$

$\Rightarrow y_0 y_2 = y_1^2, \quad y_0 y_3 = y_2 \tilde{y}_2$

$y_0 y_4 = y_1 y_3, \quad y_1 y_4 = y_2 y_3$

$\tilde{y}_2 y_4 = y_3^2$

toric picture



$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,2))$

$Y \hookrightarrow \mathcal{Y}$

$t = 0 \quad \underline{\mathcal{Y}_0 = Y}$ by ——

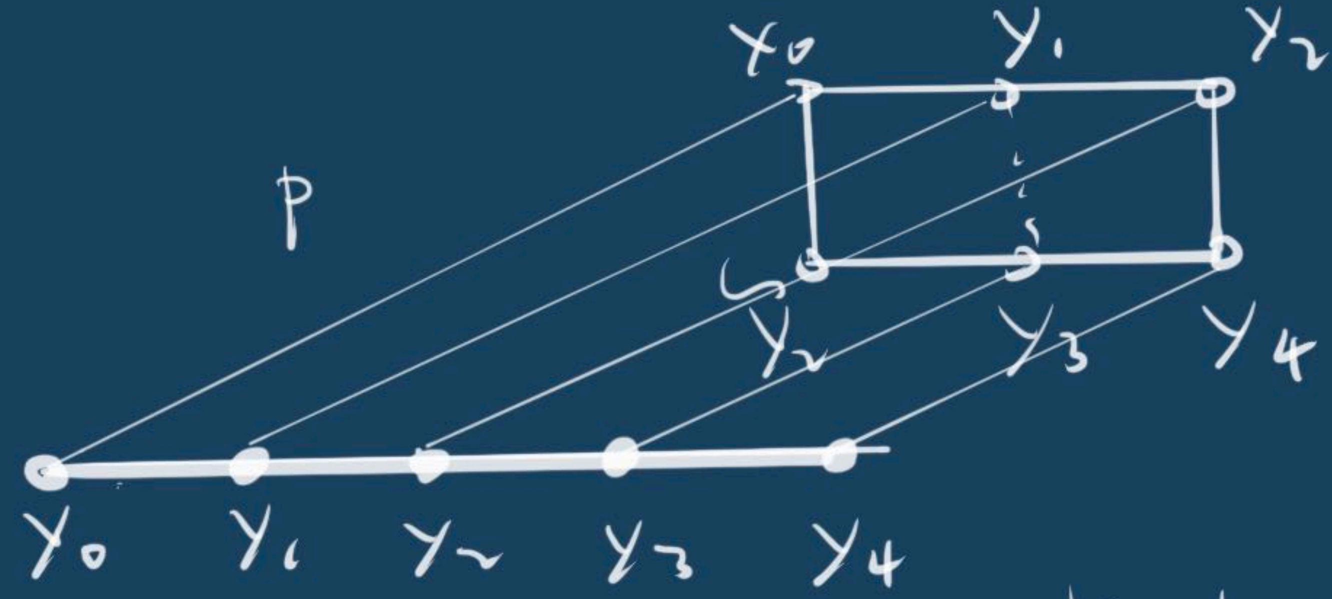
$\Downarrow$

$\tilde{y}_2, y_2$

$\downarrow$

same pt.

$M_Y \leftarrow M_{\mathcal{Y}}$



$M_{\mathcal{Y}} = \mathbb{Z}^2$
$\cap$
$\widetilde{M}_{\mathcal{Y}}$

$M_Y = \mathbb{Z}$
$\cap$
$\widetilde{M}_Y$

slicely

$\widetilde{M}_{\mathcal{Y}} \to \widetilde{M}_Y$

$\tilde{p}: \mathbb{Z}^3 \to \mathbb{Z}^2$

given by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

On the dual side

$i: \mathbb{Z}^2 \to \mathbb{Z}^3$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$

y_2 , $\tilde{y}_2$

$\Downarrow$ $\Downarrow$

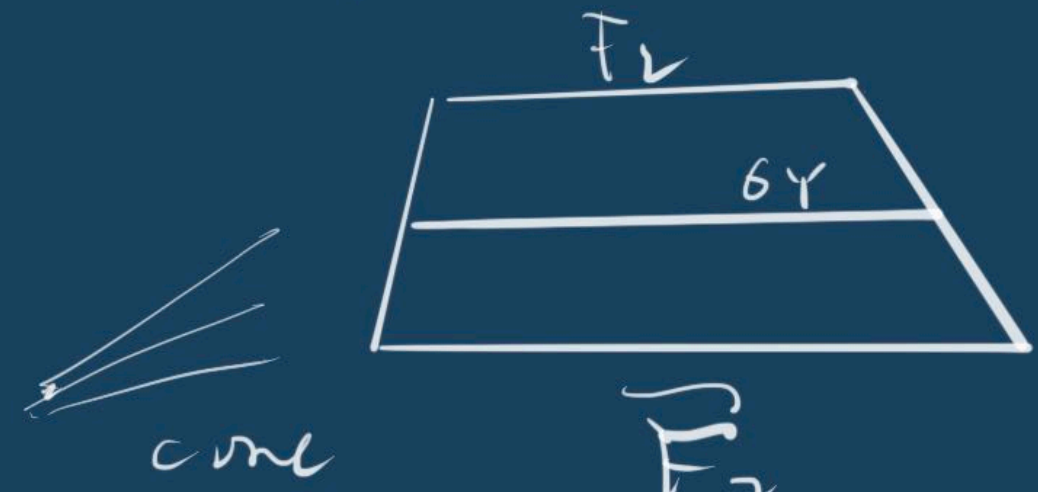
facet facet

F_2 $\tilde{F}_2$

$\subset$ $\supset$

parallel.

slicely : G_Y is in the middle:

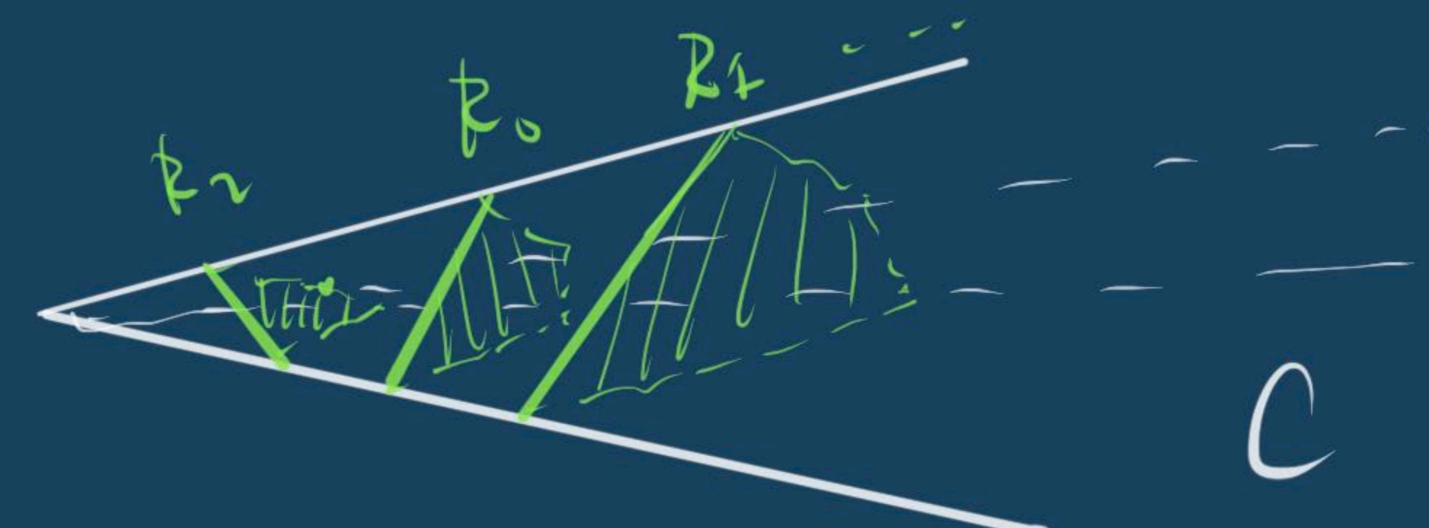


General theory :

Def: A deformation element of size m is a tuple

$(R_0, \dots, R_m ; C)$ s.t.

- $C \subseteq N_{\mathbb{R}}$ is a strongly convex polyhedral cone.
- R_i's are rational polyhedra w/ cone C.



The deform. element is admissible if $\forall t \in C^\vee \subseteq M_{\mathbb{R}}$

at least m of $m+1$ faces

$F(R_i, t) := \{ a \in R_i : \langle a, t \rangle = \min \langle R_i, t \rangle \}$

contain $\mathbb{Z}$-pts.

Goal: given $(R_0, \dots, R_m; C)$

how? $\Downarrow$

toric var's X, Y s.t.

$Y \subseteq X$, Y is defined by a toric regular seq.

Let $Q := R_0 + \dots + R_m \subseteq N_{\mathbb{R}}$.

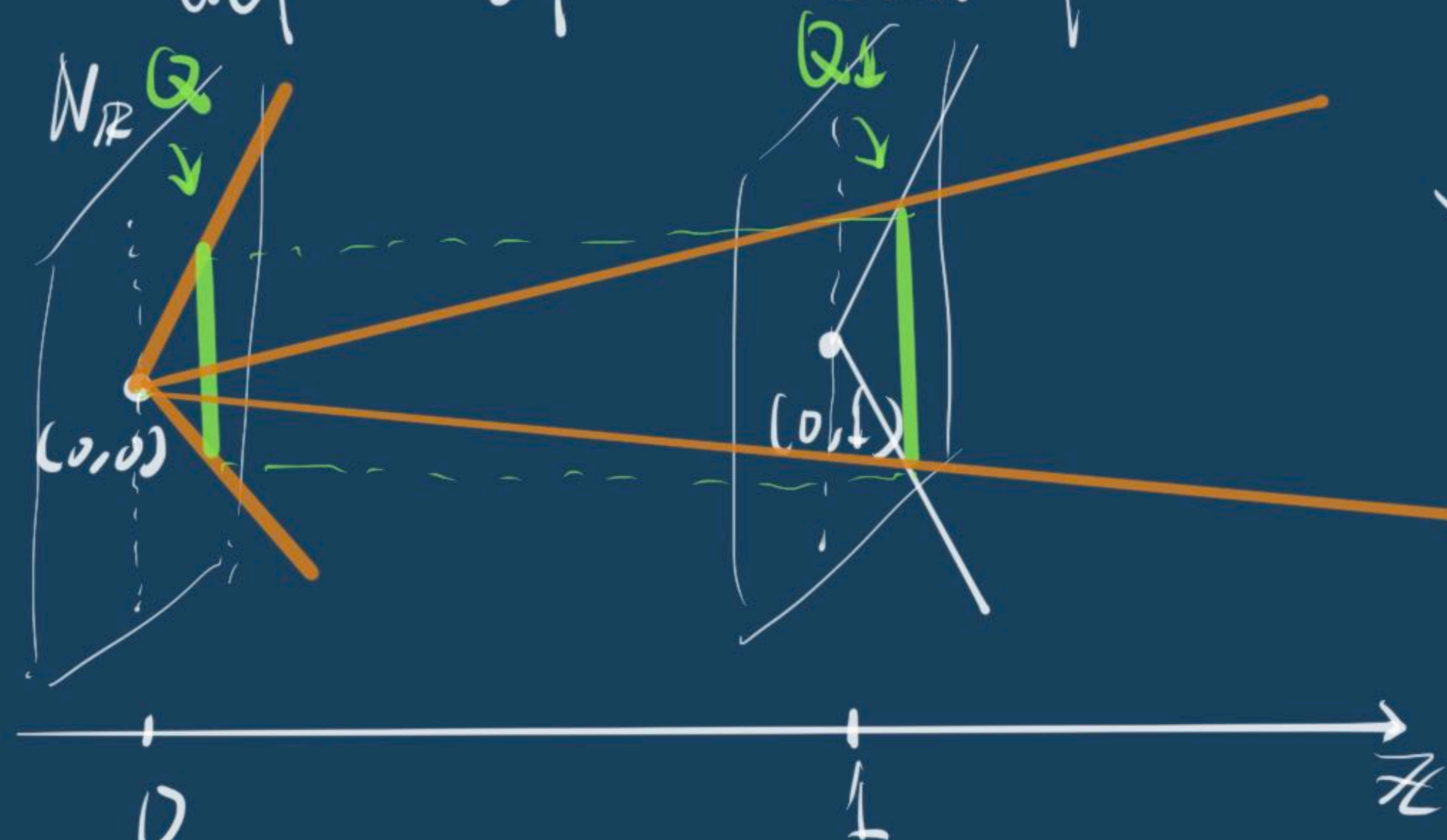
 Minkowski sum

$\tilde{N} = \mathbb{Z} \oplus N$, $\tilde{M} = \mathbb{Z} \oplus M$, $M = N^\vee$

$\psi : N_{\mathbb{R}} \hookrightarrow \tilde{N}_{\mathbb{R}}$ $\qquad Q_1 := \psi(Q)$

$a \mapsto (1, a)$

def: $\sigma_Y := \mathrm{Conv}\{\{0\} \times C \cup \mathbb{R}_{\geq 0} Q_1\}$



$\mathbb{Z} \otimes \mathbb{R} = \mathbb{R}$

$Y := \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap \tilde{M}]$

dim $n+1$

$n = \mathrm{rank}\, N$.

To define X, put R_i's in // affine planes of a vector space large enough.

Instead of add $\mathbb{Z}$, we add $\mathbb{Z}^{m+1}$.

$N' := \mathbb{Z}^{m+1} \oplus N \rightsquigarrow N'_{\mathbb{R}}, M'_{\mathbb{R}}, \dots$

$\Phi_{\mathbb{R}} : N'_{\mathbb{R}} \to \mathbb{Z}^{m+1} \otimes \mathbb{R} = \mathbb{R}^{m+1}$

proj to the 1-st factor

$\phi_i : N \hookrightarrow N' = \mathbb{Z}^{m+1} \oplus N$

$a \mapsto (e_i, a)$

$e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 at the i-th)

$\tilde{R}_i := \phi_i(R_i) \ (\subseteq \Phi_{\mathbb{R}}^{-1}(e_i))$

$P := \mathrm{Conv}(\cup_{i=0}^{m} \tilde{R}_i)$

$\sigma_X := \mathrm{Conv}(\{0\} \times C \cup \mathbb{R}_{\geq 0} P)$

$X = \mathrm{Spec}\, \mathbb{C}[\sigma_X^\vee \cap M']$

$\dim X = n + m + 1$

get: Y, X.

$p_i : \mathbb{Z}^{m+1} \to \mathbb{Z}$

i-th proj

$$\begin{tikzcd}
\varphi^i := p_i \circ \Phi : N' \arrow[rr] & & \mathbb{Z} \\
\mathbb{Z}^{m+1} \oplus N \arrow[u, phantom, "\shortparallel"] \arrow[dr, "\Phi"'] & & \\
& \mathbb{Z}^{m+1} \arrow[uur, "p_i"'] &
\end{tikzcd}$$

$\cap$

$(N')^\vee = M'$

also $\varphi^i \in \sigma_X^\vee$.

$\eta : \widetilde{N} \hookrightarrow N'$

$(l, a) \longmapsto (l, l, \ldots, l, a)$

η satisfies

i). $\widetilde{N} = N' \cap \left(\cap_{i,j} (\varphi^i - \varphi^j)^\perp \right)$

$= N' \cap \left(\cap_{i=1}^m (\varphi^0 - \varphi^i)^\perp \right)$

ii). $\sigma_Y = \sigma_X \cap \widetilde{N}_\mathbb{R}$

ii) gives $Y \hookrightarrow X$ closed emb.

as the special fiber def.

by

$(*)$: $x^{\varphi^0} - x^{\varphi^1}, \ldots, x^{\varphi^0} - x^{\varphi^m}$

Thm (Altmann).

a). $(R_0, \ldots, R_m; C)$ admissible

$\Downarrow$ by the construction above

$Y \hookrightarrow X$ and $(*)$

$(*)$ is a toric reg. seq.

b). Any toric reg. seq. can be obtained in this way from some admissible deform. element.