Recall: length estimation.

$X = X_\Sigma$ proj toric, D is a toric div. coef in $[0,1]$

$K_X + D$ is $\mathbb{Q}$-Cartier

$\Rightarrow$ $\forall$ extr. ray $R = \mathbb{R}_{\geq 0}[C]$

$\exists$ $(n-1)$-dim cone $\tau \in \Sigma$, $V(\tau) \in R \setminus \{0\}$ s.t.

$-(K_X + D) \cdot V(\tau) \leq n+1$. $\quad n = \dim X$.

Moreover: if $\begin{cases} X = \mathbb{P}^n \\ \sum d_i < 1 \end{cases}$ is not true

$\Rightarrow$ $n+1$ can be replaced by n.

last time:

Σ simplicial, $\rho(X) = 1$

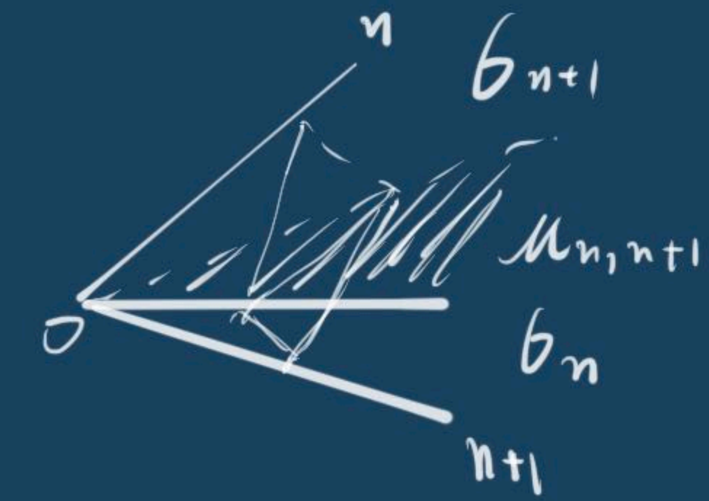
$\Sigma(1) = \{ \rho_1, \cdots \rho_{n+1} \}$

$\uparrow \qquad \uparrow$

$v_1 \cdots v_{n+1}$ primitive vectors

$\sigma_i = \langle v_1, \cdots, v_{i-1}, \widehat{v_i}, v_{i+1}, \cdots, v_{n+1} \rangle$ dim n.

$\mu_{i,j} = \sigma_i \cap \sigma_j$ dim $(n-1)$



We computed the intersections:

assumption

$\sum_{i=1}^{n+1} a_i v_i = 0$, $a_1 \leq a_2 \leq \cdots \leq a_{n+1}$

$\cdot\ V(v_{n+1}) V(\mu_{n,n+1}) = \dfrac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)}$

$\cdot\ V(v_i) V(\mu_{n,n+1}) = \dfrac{a_i}{a_{n+1}} \cdot \dfrac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)}$

Recall: $-K_X = \sum_{\rho \in \Sigma(1)} D_\rho$, $\quad D_\rho = V(v_i)$

$\Rightarrow$ $-K_X \cdot V(\mu_{n,n+1}) = \sum_{i=1}^{n+1} V(v_i) \cdot V(\mu_{n,n+1})$

$= \dfrac{a_1 + \cdots + a_{n+1}}{a_{n+1}} \cdot \underset{\sim}{\dfrac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)}}$

$\leq n+1$

And $-K_X \cdot V(\mu_{n,n+1}) = n+1$

$\Downarrow$

$\begin{cases} \mathrm{mult}(\sigma_n) = \mathrm{mult}(\mu_{n,n+1}) \\ a_i \text{ are the same.} \end{cases}$

Prop. X toric, proj, $\mathbb{Q}$-factorial, $X \not\cong \mathbb{P}^n$

$\Rightarrow \exists\ (l,m)$ s.t. $-K_X \cdot V(\mu_{l,m}) \le n$

Pf: Otherwise

$$-K_X \cdot V(\mu_{k,n+1}) = \frac{a_1+\cdots+a_{n+1}}{a_{n+1}} \cdot \frac{\mathrm{mult}(\mu_{k,n+1})}{\mathrm{mult}(\sigma_k)} > n$$

$$\Rightarrow (n+1)\,a_{n+1} \ge a_1+\cdots+a_{n+1} > \frac{\mathrm{mult}(\sigma_k)}{\mathrm{mult}(\mu_{k,n+1})} \cdot n\, a_{n+1}$$

$$\underbrace{\frac{n+1}{n}}\cdot a_{n+1} > \underbrace{\frac{\mathrm{mult}(\sigma_k)}{\mathrm{mult}(\mu_{k,n+1})}}\cdot a_{n+1}$$

$$\Rightarrow \frac{\mathrm{mult}(\sigma_k)}{\mathrm{mult}(\mu_{k,n+1})} = 1$$

$$1 = \frac{\mathrm{mult}(\mu_{k,n+1})}{\mathrm{mult}(\sigma_k)} = \frac{a_{n+1}}{a_k} \cdot \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_{n+1})}$$

$$\frac{a_{n+1}}{a_k} = \frac{\mathrm{mult}(\sigma_n)}{\mathrm{mult}(\mu_{n,n+1})} \in \mathbb{Z}$$

$\Rightarrow a_k \mid a_{n+1}$ for all k.

If $a_1 \ne a_{n+1} \Rightarrow a_2 \ne a_{n+1}$, otherwise

$a_1 v_1 = a_{n+1}(v_2 + \cdots + v_{n+1})$

not primitive. $\rightarrow v_1 = \frac{a_{n+1}}{a_1}(\cdots)$.

$a_1 \le \frac{1}{2} a_{n+1}$, $a_2 \le \frac{1}{2} a_{n+1}$

$a_1 + a_2 \le a_{n+1} \Rightarrow \sum_{i=1}^{n+1} a_i > n\, a_{n+1}$ impossible

$\Rightarrow a_1 = a_{n+1}$, a_i's are all equal.

we assumed $\gcd(a_i\text{'s}) = 1$

$\Rightarrow a_i = 1$, $i = 1, \dots, n$.

If $-K_X \cdot V(\mu_{l,m}) > n$

$$-K_X \cdot V(\mu_{l,m}) = -\sum_{k=1}^{n+1} V(v_k) \cdot V(\mu_{l,m})$$

$$= \sum_{k=1}^{n+1} \frac{a_k}{a_m} V(v_m) \cdot V(\mu_{l,m})$$

$$= (n+1) \cdot V(v_m) \cdot V(\mu_{l,m})$$

$$= (n+1) \cdot \underbrace{\frac{\mathrm{mult}(\mu_{l,m})}{\mathrm{mult}(\sigma_l)}}_{\le 1}$$

$\Rightarrow -K_X \cdot V(\mu_{l,m}) = n+1$

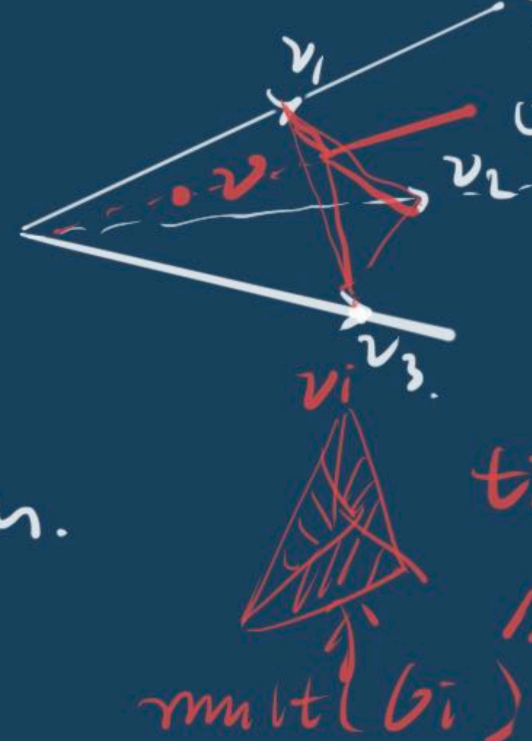
$\Rightarrow \mathrm{mult}(\sigma_l) = \mathrm{mult}(\mu_{l,m})$

$\Rightarrow \mathrm{mult}(\sigma_l) = 1$ for all l.

$\Rightarrow X \cong \mathbb{P}^n$

Contradiction.

$\exists v = \sum t_i v_i$, $0 < t_i < 1$



$X = X_\Sigma$, Σ simplicial.

R: extr. ray

$\|$

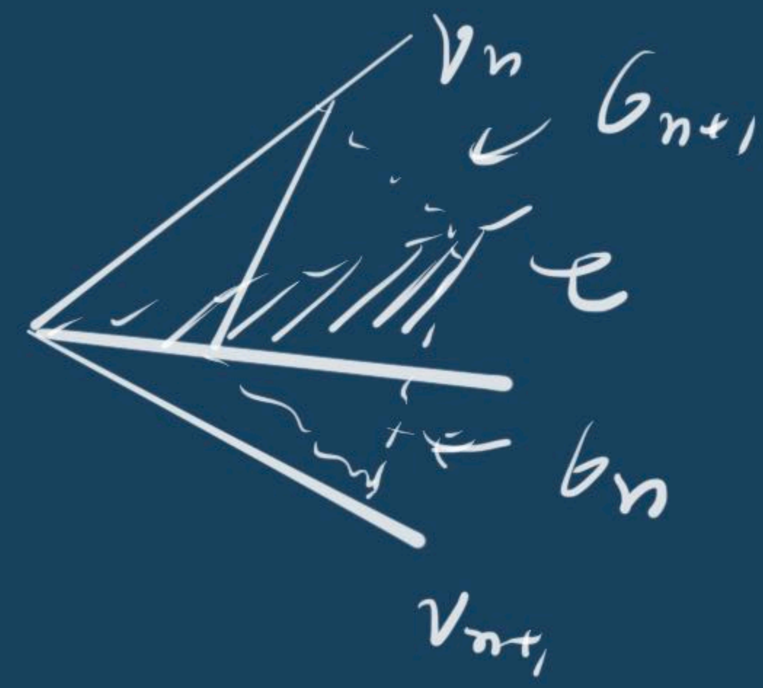
$\mathbb{R}_{\geq 0}[C] = \mathbb{R}_{\geq 0}[V(\tau)]$

$(\mu_{n,n+1} =)\ \tau$: $(n-1)$-dim cone.

$\| \quad \langle v_1, \cdots, v_{n-1} \rangle$

$\sigma_n \supset \tau \subset \sigma_{n+1}$ full dim cones

add v_{n+1} add v_n



write $\sum_{i=1}^{n+1} a_i v_i = 0$, assume $a_i \nearrow$, $a_{n+1} = 1$.

since v_n, v_{n+1} are on the opp. sides of τ $\Downarrow$ $a_n = 1$

$\exists\ \alpha, \beta$ s.t.

$$\begin{cases} a_i < 0, & 1 \leq i \leq \alpha & \mathrm{I} \\ a_i = 0, & \alpha+1 \leq i \leq \beta & \mathrm{II} \\ a_i > 0, & \beta+1 \leq i \leq n+1 & \mathrm{III} \end{cases}$$

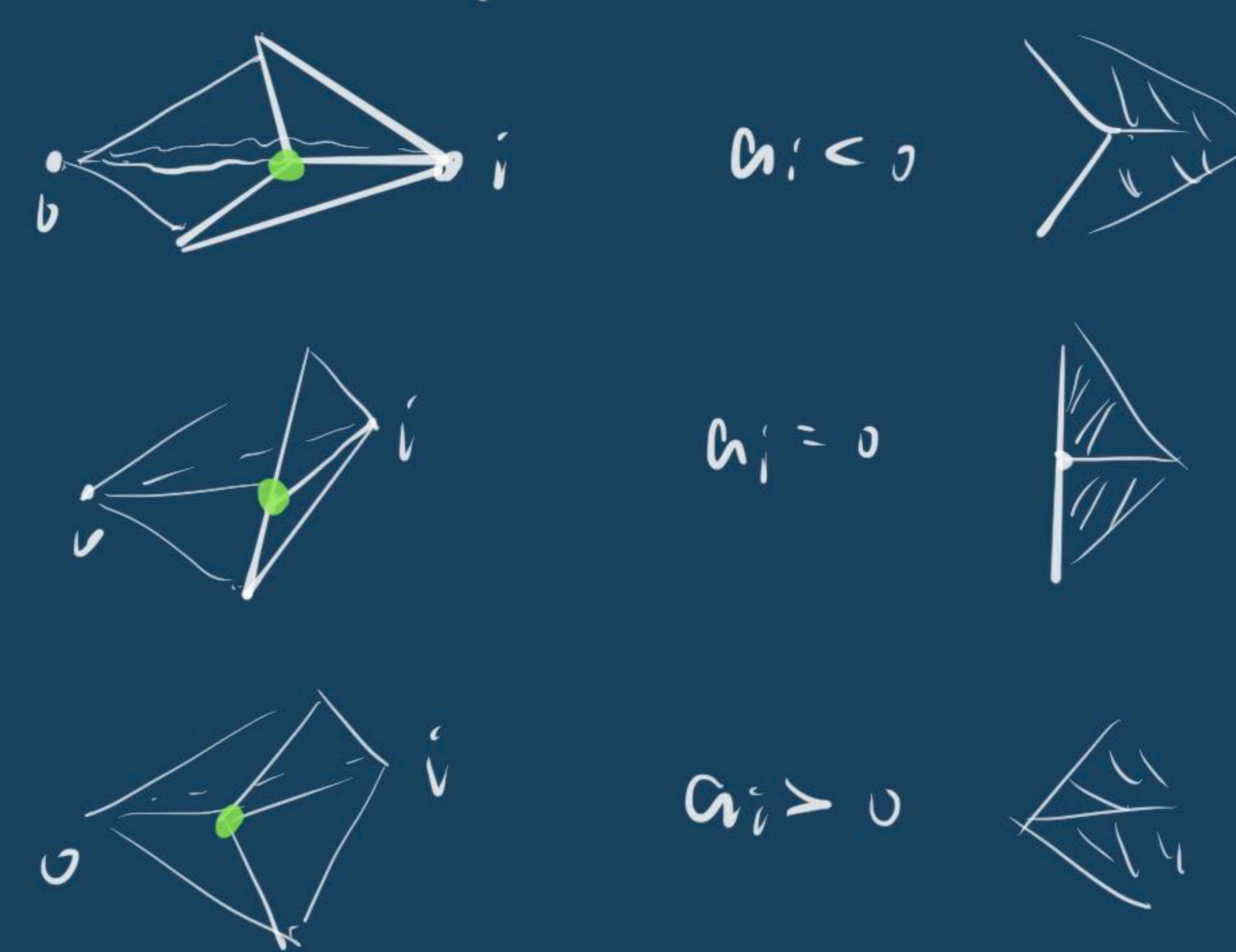
Rmk: geo: i in I/III $\Leftrightarrow$ $D_i \cdot C \ {<0}/{>0}$

$\left(V(v_i) \cdot V(\tau) \ {<0}/{>0} \right)$

α describes the contraction type.

$$\alpha \begin{cases} = 0 & \text{Fano-Mori fibration} \\ = 1 & \text{div.} \\ \geq 2 & \text{flipping / small.} \end{cases}$$

Comb. : shape of $\sigma_n \cup \sigma_{n+1}$



convexity along $\langle v_1, \dots, \hat{v_i}, \dots, v_{n-1} \rangle$

Contract R:

$$\begin{array}{ccc} X & & A \\ \varphi_R \downarrow & & \downarrow \\ Y & & B \end{array} \quad \text{non-iso loci}$$

$\varphi_R(C) = pt$

Reid's result: $A \supseteq$ cones gen. by $v_1, \dots, v_\alpha$

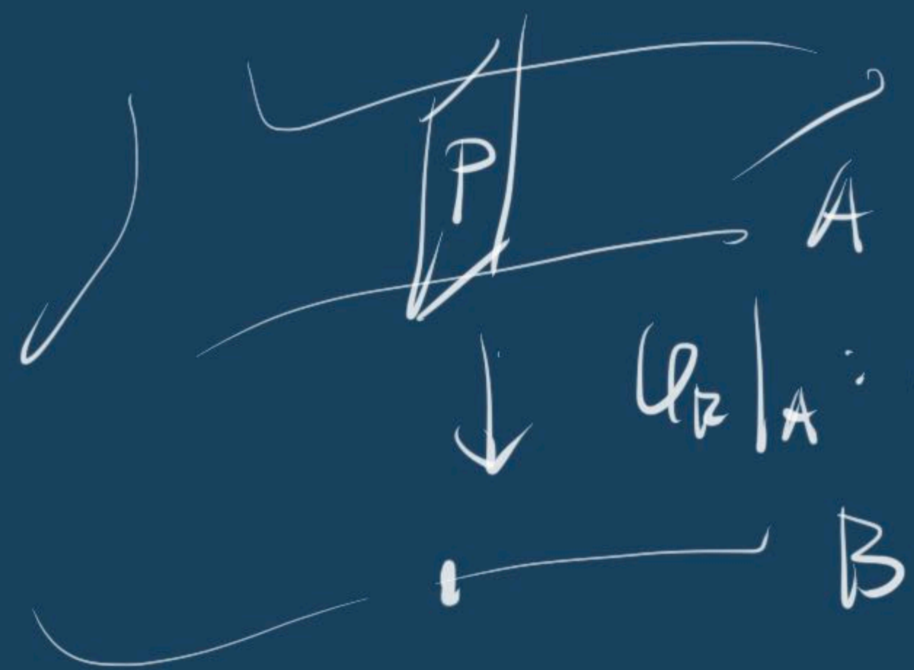
$\varphi_R \downarrow$

$B \supseteq$ cones gen. by $v_1, \dots, v_\alpha, v_{\beta+1}, \dots, v_{n+1}$

$\dim A = n - \alpha$

$\dim B = \beta - \alpha$

$\underbrace{\sigma_P = \langle v_1, \dots, v_\beta \rangle}$



$P = V(\sigma_P)$: $\cdot$ $\mathbb{Q}$-factorial

$\cdot$ Fano

$\cdot$ $\rho(P) = 1$

$\cdot$ dim $n - \beta$

$\cdot$ $\varphi_R(P) = pt.$

$\varphi_R|_A$: all fibers are dim$(n-\beta)$

or

Rmk: Q: What are we doing in the toric picture when we contract R?

A: Break walls.

Reid: $\Sigma = \Sigma_n + \Sigma_{n+1}$

$\displaystyle \parallel \bigcup_{i=\beta+1}^{n+1} \sigma_i$ Why?

Because: $x = \sum x_i v_i \in \sigma$, $x_i \geq 0$

find j s.t. $j \geq \beta+1$, x_j / a_j min.

$x = \sum x_i v_i - \frac{x_j}{a_j} \underline{\sum a_i v_i}$

$= x_1 v_1 + \cdots + \cancel{x_j v_j} + \cdots + x_{n+1} v_{n+1}$

$- \frac{x_j a_1}{a_j} v_1 + \cdots + \cancel{x_j v_j} + \cdots + \frac{x_j a_{n+1}}{a_j} v_{n+1}$

$\in \sigma_j$

$[V(\mu_{l,m})] \in R.$

$\parallel$

$\underline{\sigma_l \cap \sigma_m}$

walls are removed by φ_R.

$P = V(\sigma_P)$, $\sigma_P = \langle v_1, \dots, v_\beta \rangle$

$K_P = -\sum_{i=\beta+1}^{n+1} V(\widetilde{\rho_i})$, $\widetilde{\rho_i} = \langle v_1, \dots, v_\beta, v_i \rangle$

$\beta+1 \le i \le n+1$

- $V(\widetilde{\rho_i}) = b_i V(v_i) \cdot V(\sigma_P)$, $\underline{b_i \in \mathbb{Z}_{>0}}$ (*).

$\widetilde{\tau}$: $(n-1)$-dim cone containing σ_P

- also: $V(v_i) \cdot V(\widetilde{\tau}) > 0 \iff i$ is in $\{\beta+1, \dots, n+1\}$.

$$K_P \cdot V(\widetilde{\tau}) = -\sum_{i=\beta+1}^{n+1} V(\widetilde{\rho_i}) \cdot V(\widetilde{\tau})$$

$$= -\Big(\sum_{i=\beta+1}^{n+1} b_i V(v_i) V(\sigma_P)\Big) \cdot V(\widetilde{\tau})$$

$$= \Big(K_X + \sum_{\text{all}} V(v_i) - \sum_{i=\beta+1}^{n+1} b_i V(v_i)\Big) \cdot V(\widetilde{\tau})$$

$$= \Big(K_X + \underbrace{\sum_{i=\beta+1}^{n+1} (1-b_i) V(v_i)}_{(*)\ \le 0} + \underbrace{\sum_{i=1}^{\beta} V(v_i)}_{\text{neg int's.}}\Big) \cdot V(\widetilde{\tau})$$

$$\le (K_X + D) \cdot V(\widetilde{\tau})$$

$$-(K_X + D) \cdot V(\widetilde{\tau}) \le -K_P V(\widetilde{\tau})$$

if $\min\{-(K_X+D) \cdot C\} > n$

$\Rightarrow \min\{-K \cdot C\} > n$

$\Rightarrow \alpha = \beta = 0.$

$\Rightarrow X = \mathbb{P}^n$

otherwise. bound is $\le n$.

X is not $\mathbb{Q}$-factorial:

Apply proj modification:

(X, D) toric pair.

X: toric proj

D: toric inv. div. coeff in $[0,1]$

$K_X + D$: $\mathbb{Q}$-Cartier.

Then $\exists f : \widetilde{X} \to X$ proj bir. toric

s.t. $\widetilde{X}$ $\mathbb{Q}$-factorial

and $K_{\widetilde{X}} + \sum \widetilde{d_i} \widetilde{D_i} = f^*(K_X + \sum d_i D_i)$

Choose R extr.

$\Downarrow$

Choose pt

$\Downarrow$ s.t. $-(K_X+D)\cdot V(\tau)$ min. (*)

$V(\tau)$

Take:
$$\begin{array}{ccc} \tilde{X} & \supseteq & V(\tilde{\tau}) \\ f\downarrow & & \downarrow \\ X & \supseteq & V(\tau) \end{array} \qquad f_* V(\tilde{\tau}) = V(\tau).$$

By previous: $V(\tilde{\tau}) = \sum a_i V(\tilde{\tau}_i)$

$a_i > 0$, $V(\tilde{\tau}_i)$ extr.

$-(K_{\tilde{X}}+\tilde{D})\cdot V(\tilde{\tau}_i) \le n$ $(\tilde{X} \neq \mathbb{P}^n)$

for all i's.

$\sum a_i f_* V(\tilde{\tau}_i) = V(\tau) \in R$

$\Rightarrow f_* V(\tilde{\tau}_i) \in R$

$\qquad \shortparallel$

So $\quad b V(\tau)$, $\quad b \ge 1$. by (*)

$-(K_X+D)\cdot V(\tau) = -\frac{1}{b}(K_{\tilde{X}}+\tilde{D})\cdot V(\tilde{\tau}_i) \le n$. □

By the same argument: rel. version

$f: X \to Y$ $\quad$ $\dim X = n$.

proj, toric $\quad$ X: $\mathbb{Q}$-Gorenstein

(K_X: $\mathbb{Q}$-Cartier)

R: extr. ray in $NE(X/Y)$

$\shortparallel$

$NE(X) \supseteq \{1 \text{ cycles } \sum a_i C_i,\ f(C_i) = pt\}$

$\varphi_R : X \to W$

(*) If φ_R is not bir.

$\Rightarrow l(R) < \max \dim \varphi_R^{-1}(w) + 1$

$l(R) = \min_{[C]\in R}(-K_X\cdot C)$, $\quad \max \dim \varphi_R^{-1}(w) = d$ $(\le n-1)$

i.e. $l(R) < d+1$.

If $d = n-1$, then

$l(R) \le n-1$ sharp.

"=" $\Rightarrow \varphi_R$ is wted bl. up of wt $(1, a, \dots, a)$

$a \in \mathbb{Z}_{>0}$

Nonvanishing Thm in toric case.

Set up: X : dim n. proj, $\mathbb{Q}$-Gorenstein toric

D : $\mathbb{Q}$-div. ample, Cartier. on X

Thm. $K_X + (n-1)D$ pseudo effective (ps.eff)

$\Downarrow$

$K_X + (n-1)D$ nef.

· If X is Gorenstein:

$H^0(X, \mathcal{O}_X(K_X + (n-1)D)) \neq 0 \Leftrightarrow$ complete linear system $|K_X + (n-1)D|$ is BFF.

Def: D is $\mathbb{R}$-div. D is ps.eff if

$D \equiv \sum d_i D_i$, $d_i \geq 0$ D_i Cartier.

$PE(X) = \{D \mid D \text{ ps.eff}\}$.

Rmk: · $Nef(X) \subseteq PE(X)$

· examples of $\mathbb{Z}$-div ps.eff not eff is tricky.

· Thm $\overline{Big(X)} = PE(X)$

Prop. X complete toric

D: $\mathbb{Q}$-Cartier.

D ps.eff

$\Downarrow$

$\exists\, m \in \mathbb{Z}_{>0}$ s.t.

$H^0(X, \mathcal{O}_X(mD)) \neq 0$

$\left(\begin{array}{c} \| \\ \kappa(X, D) \end{array} \geq 0 \right)$

Rmk: There is a general version of the prop. above due to Cascini-Hacon-Mustata-Schwede.