Last time: para. 4 pts on $\mathbb{P}^1 / \mathrm{aut}(\mathbb{P}^1)$

$Gr(2,4)/\!/_{ch} T$ (in $\mathbb{P}^5 /\!/_{ch} T$)

should be $\mathbb{P}^1$

the 2-nd-ary polytope.

On $\mathbb{P}^1$: fix 3 pts. (*) (on $\mathbb{P}^{k-1}$, can always fix $k+2$ hyperplanes)

0, 1, ∞

the 4-th pt is free.

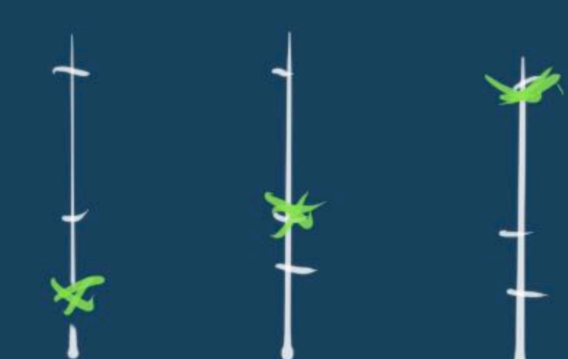
$\underline{M^0_{2\times 4}}$: $\begin{pmatrix} * & * & * & * \\ * & * & * & * \end{pmatrix}$

(*) means: $\begin{pmatrix} 1 & 0 & 1 & * \\ 0 & 1 & 1 & * \end{pmatrix}$ no 0-Pl. coord.

$Gr^0(2,4)/T = \mathbb{P}^1 \setminus$ 3 pts

$\cap$

$Gr(2,4)/\!/_{ch} T = \mathbb{P}^1$

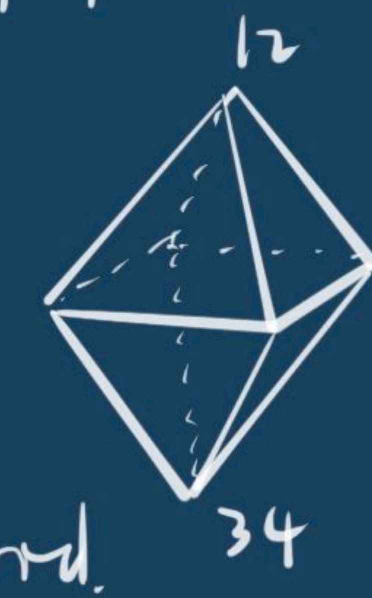


no quite.

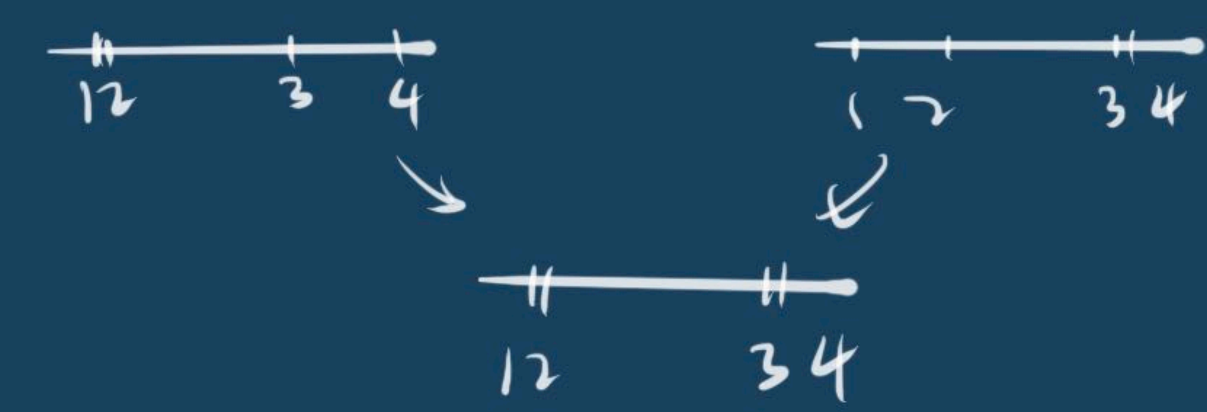
$Gr(2,4)/\!/_{ch} T$

Correct picture:

$Gr(2,4)/\!/_{ch} T$



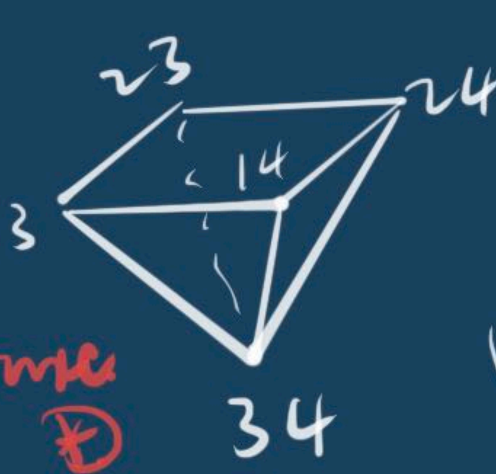
fibers over the boundary (non-generic)



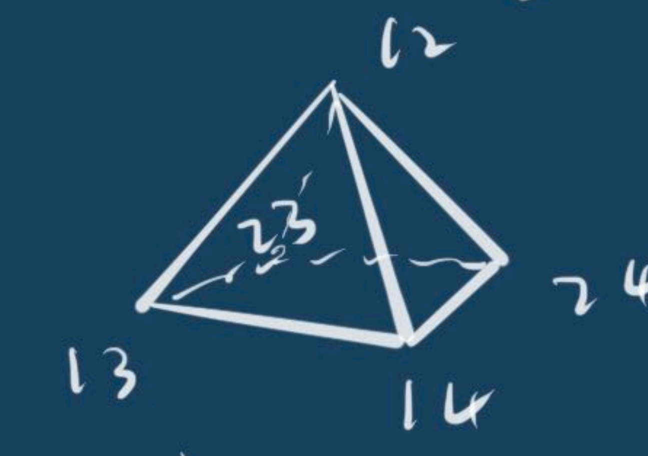
Polytope picture:

12 overlapped, remove (1,1,0,0)

similarly:



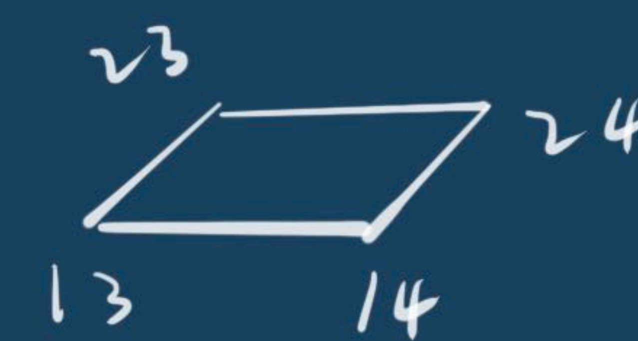
picture (*)



remove:

(1,1,0,0)

(0,0,1,1)



Q: Do we get all decompositions of

?

A: No!

e.g. we did not see Δ'tions.

Q: Which decompositions are correct?

- Def: A matroid is a pair $(E, \mathcal{B})$

  E: a finite set

  $\phi \neq \mathcal{B}$ a set in 2^E, called bases

  s.t. · $\forall I, J \in \mathcal{B}$, $i \in I - J$

  $\Rightarrow \exists\, j \in J - I$ s.t.

  $(I - \{i\}) \cup \{j\} \in \mathcal{B}$;

  · no $I \subsetneq J$ in $\mathcal{B}$.

Rmk: this is a generalization of vector spaces.

e.g. $E = k^4$, $\mathcal{B} = \{\{e_i, e_j\},\ 1 \le i < j \le 4\}$. ④

Each matroid $(E, \mathcal{B}) \rightsquigarrow$ a polytope.

how? $\mathrm{Conv}\{ e_I = \sum_{i \in I} e_i \mid I \text{ runs over } \mathcal{B} \}$

e.g. the polytope associated to ④ is $\Delta_{2,4}$.

Thm (Gelfand - Goresky - Macpherson - Serganova)

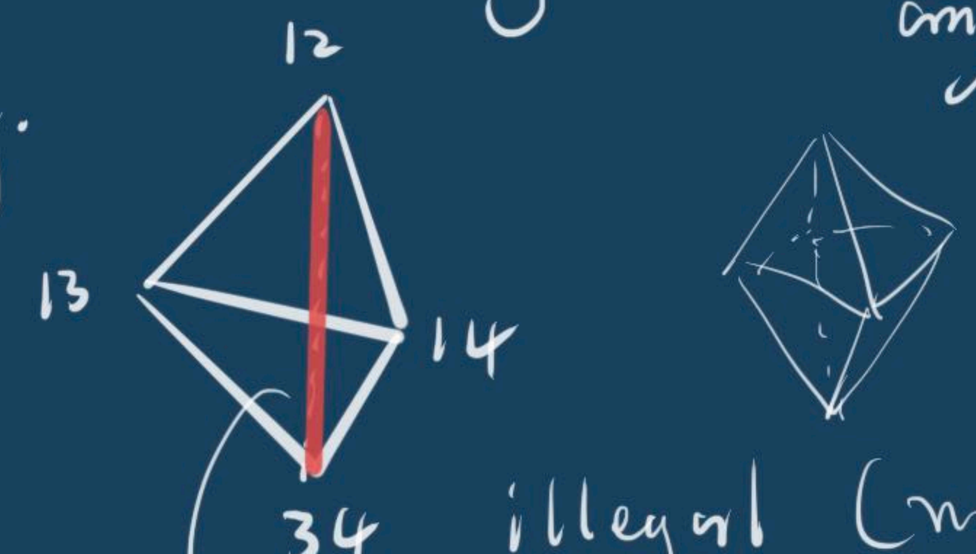
$\Delta_{k,n} \supseteq Q$: a lattice polytope

Q comes from a matroid

$\Updownarrow$

- $\mathrm{Vert}(Q) \subseteq \mathrm{Vert}(\Delta_{k,n})$
- $\mathrm{Edge}(Q) \ni \ell$ (any) $/\!/ \ e_i - e_j$ for some i, j.

e.g.



illegal (not matroid)

$e_1 + e_2 - e_3 - e_4 \not\parallel e_i - e_j$

Thm. In $Gr(k,n)/\!/_{ch} T$,

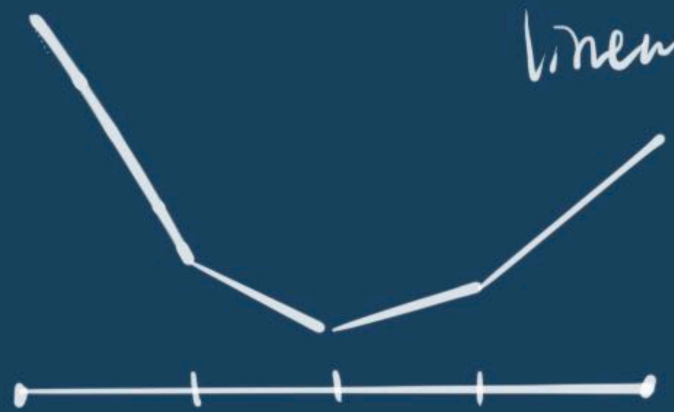
degenerations

$\Updownarrow$

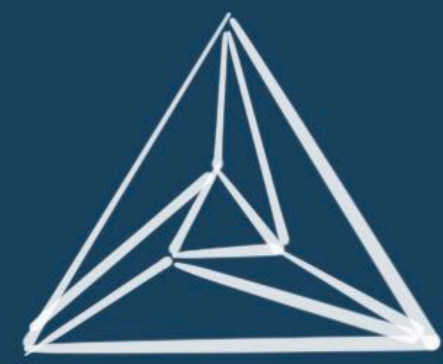
- regular
- each subpoly. is matroid.

decompositions of $\Delta_{k,n}$

regular: the decomposition comes from a piecewise convex linear function (ht function)



non-regular:



To see the Chow quotient of HA, need computation results for the right decomp. of $\Delta_{k,n}$

$\Delta_{2,n}$ easy ($\leftrightarrow$ graphs)

$\Delta_{3,n}$ $n \lesssim 10$ or 11 ?

$\Delta_{4,n}$ $n=8$ is still open.

$(\Delta_{k,n} \cong \Delta_{n-k,n})$

Summary: $Gr(k,n) \xrightarrow{Pl.} \mathbb{P}^N \qquad N=\binom{n}{k}-1$

$Gr(k,n)/\!/_{ch}T$ $\cup$ is irr. regular matroid

$\cap \downarrow$

$Gr(k,n)/\!/_{lim}T$ matroid

$\cap \downarrow$

$\mathbb{P}^N/\!/_{lim}T$ all

$\downarrow$ Chow quotient

limit quotient.

Fact: $X/\!/_{ch}G \xrightarrow{\varphi} X/\!/_{lim}G$ over the main cpt.

(Thm (Yi Hu): φ is bir. homeo, under some condition)

Rmk: $k=2$, all decomp.'s are regular.

$Gr(2,n)/\!/_{ch}T \cong Gr(2,n)/\!/_{lim}T$

$\cong$ Kapranov

$\overline{M}_{0,n}$

Minimal Model Program (MMP)

(Mori)

Goal: classify alg. var's up to bir equ.

e.g. in dim 2:

S : surf.

$$\begin{array}{ccc} & & S \\ & \nearrow^{\text{res.}} & \\ \tilde{S} & \xrightarrow{\text{contract } (-1)\text{-curves}} & S_{min} \end{array}$$

Here min means 1). no (-1)-curves

2). smooth

Rmk: 1) $S \supseteq C$ s.t. $C^2 = -1$

$(K_S + C)\cdot C = K_C \Rightarrow K_S \cdot C = -1 < 0$

$K_S \cdot C - 1 = -2$

K_S is not nef.

2). in dim 2. smooth = terminal

$\mathbb{C}^2 /$ finite subgp of $SL_2(\mathbb{C})$ = canonical

" " $GL_2(\mathbb{C})$ = l.t

Classified by Kawamata / Alexeev = l.c

In higher dim. min model means $\begin{cases} K_X \text{ nef} \\ \text{terminal sing.} \end{cases}$

e.g. if insist on smooth (Iitaka)

$$\begin{array}{ccc} A_3 & \text{ab. 3-fold} & \\ \downarrow {/\mathbb{Z}/2\mathbb{Z}} & & X \\ A_3/\iota & \xleftarrow{\text{res}} & \end{array}$$

$K_{X_{min}}$ nef

$\Updownarrow$ conflicts

X_{min} smooth

. in dim 2 : S_{min} is unique

but dim ≥ 3, this is not true.

e.g. sm. general type

$\#\{\text{min model}\} < \infty$

they are connected by flips.

For toric var's.

X_0: proj $\mathbb{Q}$-factorial toric var.

D_0: $\mathbb{Q}$-div.

$(X_0, D_0) \dashrightarrow \dashrightarrow (X_i, D_i) \dashrightarrow \dashrightarrow (X_{min}, D_{min})$

X_i: $\mathbb{Q}$-factorial, D_i: $\mathbb{Q}$-div on X_i

From i to $i+1$

(*) if $K_{X_i} + D_i$ nef done.

otherwise: $\exists R$ extr. ray in $\overline{NE}(X_i)$

$R \cdot (K_{X_i} + D_i) < 0$

contract R:

$\varphi_R : X_i \to Y_i$

Case 1: $\dim Y_i < \dim X_i$ (Fano-Mor: fibration)

$X_{i+1} := Y_i$, $D_{i+1} := \varphi_{R*} D_i$

go back to (*)

Case 2: φ_R is divisorial.

Case 3: φ_R is small

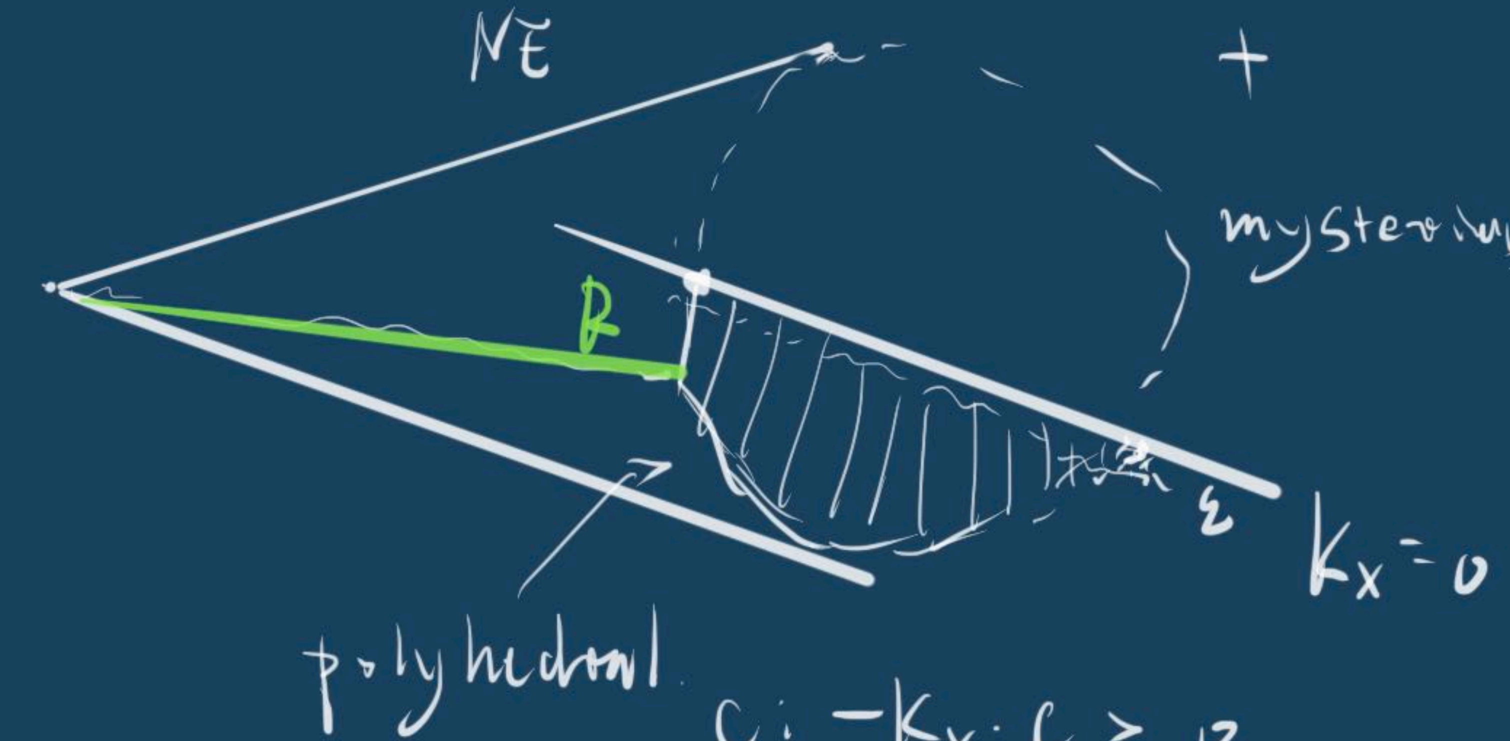
need log flip:

$$\begin{tikzcd} X_i \arrow[rr, dashed] \arrow[dr, "\varphi_R"'] & & X_i^+ \arrow[dl] \\ & Y_i & \end{tikzcd}$$

Rmk: case 1: dim $\searrow$ finitely many steps.

case 2: Picard rank $\searrow$

case 3: You did this last week.



C: $-K_X \cdot C > 0$

Cone Thm for Toric Var's.

$X = X_\Sigma$ proj toric (not necessarily $\mathbb{Q}$-factorial)

$N_1 : \dfrac{\mathbb{R}\text{-coeff } 1\text{-cycles}}{\equiv_{num}}$

$NE := \sum \mathbb{R}_{\geq 0} [C]$ which the span of $\underbrace{\text{finitely many}}$ extr. rays
all effective 1-cycles
$\cap$
N_1

$\sim_{lin} \Rightarrow \sim_{alg} \Rightarrow \equiv_{num}$.

Thm. if $K_X + D$ is $\mathbb{Q}$-Cartier
where $D = \sum d_i D_i$, $d_i \in [0,1]$
D_i : toric div

$\Rightarrow$ $\forall$ extr. ray $\mathbb{R}_{\geq 0}[C]$
$\exists$ $(n-1)$-dim cone $\tau \in \Sigma(n-1)$
$V(\tau) \in \mathbb{R}_{\geq 0}[C]$ s.t.
$-(K_X + D) \cdot V(\tau) \in n+1$

Moreover, if $X = \mathbb{P}^n$ not satisfied
$\sum d_i < 1$

$\Rightarrow$ boundary can be shrinked to n.

Tools might be useful:

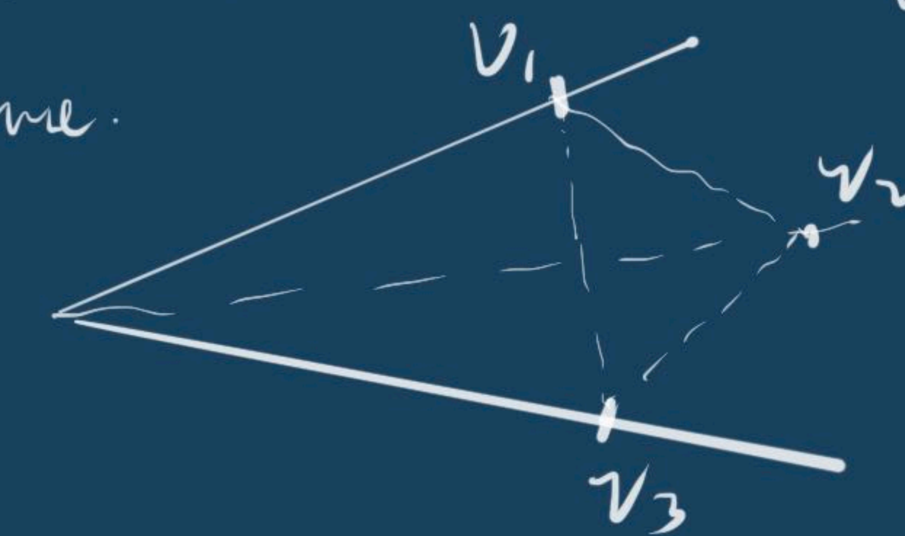
σ: full dim cone in $\Sigma(n)$. $\subseteq N_{\mathbb{R}}$

$N_\sigma := (\sigma \cap N) + ((-\sigma) \cap N)$

Assume: σ simplicial

$\{ v_i, i = 1, \cdots, n \}$ primitive vectors of $\sigma(1)$

Picture.



$\sum \mathbb{Z} v_i$ gives a sublattice N_σ'

def: $\text{mult}(\sigma) := [N_\sigma : N_\sigma']$

Note: $\text{mult}(\sigma) = 1 \Leftrightarrow X_\sigma$ is sm.

e.g. Σ: complete fan.

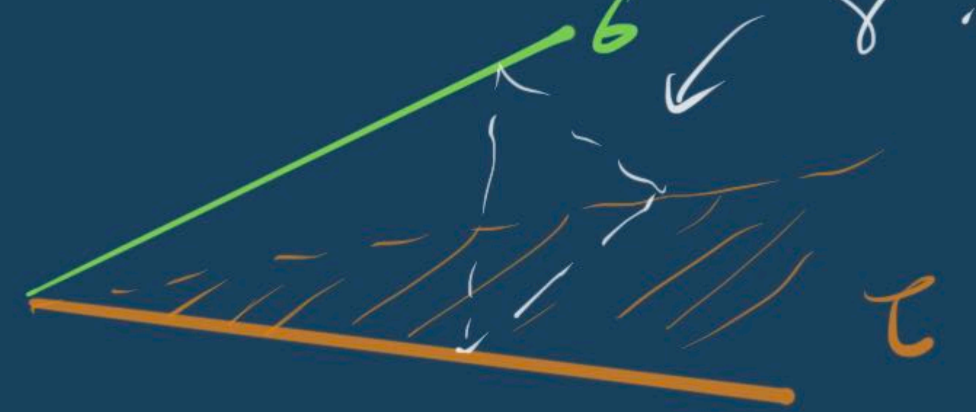
$\text{mult}(\sigma) = 1$ for all $\sigma \in \Sigma(n)$ $\Leftrightarrow X_\Sigma \cong \mathbb{P}^n$

How to compute intersections?

Formula: Σ : simplicial ($\Leftrightarrow X_\Sigma$ $\mathbb{Q}$-fact.)

$\sigma, \tau \in \Sigma$ span γ $\quad \dim \sigma + \dim \tau = \dim \gamma$

γ : dim 3 cone



$$V(\sigma) \cdot V(\tau) = \frac{\mathrm{mult}(\sigma) \cdot \mathrm{mult}(\tau)}{\mathrm{mult}(\gamma)} \cdot V(\gamma)$$

Rmk: if σ, τ shares no cone, then $V(\sigma) V(\tau) = 0$.

Assume: $\rho(X_\Sigma) = 1$ (recall $\rho = |\Sigma(1)| - \mathrm{rank}\, N$).

primitive vectors: $v_1, \cdots, v_{n+1}$

$\sigma_i :=$ Cone gen by $v_1, \cdots, v_{i-1}, \widehat{v_i}, v_{i+1}, \cdots, v_{n+1}$
$= \langle v_1, \cdots, \widehat{v_i}, \cdots v_{n+1} \rangle$

let $\mu_{i,j} = \sigma_i \cap \sigma_j$
dim $(n-1)$

WLOG: after reordering v_i's
may write
$$\sum_{i=1}^{n+1} a_i v_i = 0 \quad w/$$
$a_1 \le a_2 \le \cdots \le a_{n+1}$, $\gcd(a_1, \cdots, a_{n+1}) = 1$
in $\mathbb{Z}_{>0}$

By the intersection formula:

$$0 < V(v_{n+1}) V(\mu_{n,n+1}) = \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)} \le 1$$
why?

$V(v_i) \cdot V(\mu_{n,n+1}) = ?$

Choose dual bases : $v_1^*, \cdots, v_n^*$ in $M_{\mathbb{Q}}$

s.t $v_i^*(v_j) = \delta_{ij}$

$1 \le i,j \le n$

$\sum_{i=1}^{n+1} a_i v_i = 0$ apply v_i^*

$$a_i \underbrace{\langle v_i^*, v_i \rangle}_{1} + a_{n+1} \langle v_i^*, v_{n+1} \rangle = 0$$

$$\Rightarrow \langle v_i^*, v_{n+1} \rangle = -\frac{a_i}{a_{n+1}}$$

$$0 \sim \mathrm{Div}(\chi^m) = \sum_{i=1}^{n+1} \langle m, v_i \rangle V(v_i)$$

choose $m = v_i^*$ in $M_{\mathbb{Q}}$

$$0 \sim_{\text{lin.}} V(v_i) + \langle v_i^*, v_{n+1} \rangle V(v_{n+1})$$

$$= V(v_i) - \frac{a_i}{a_{n+1}} V(v_{n+1})$$

$$V(v_i) \sim \frac{a_i}{a_{n+1}} V(v_{n+1})$$

$$V(v_i) \cdot V(\mu_{n,n+1}) = \frac{a_i}{a_{n+1}} V(v_{n+1}) \cdot V(\mu_{n,n+1}).$$