GIT quotient: can be used in moduli construction

problems: 1). not canonical

dep on a linearization $G \curvearrowright L$

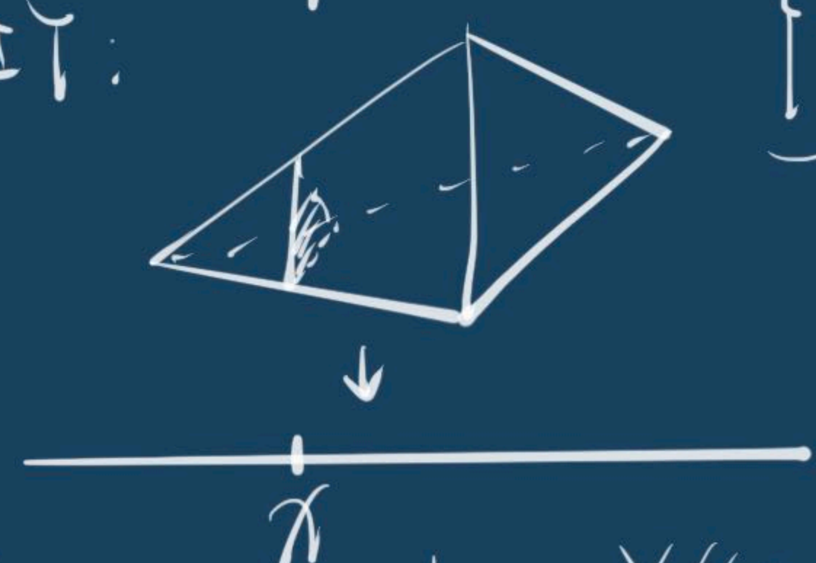
2). not modular / geometric

expect a flat π

Today: 1) will be fixed

2) restricted to some special moduli problem

Recall: for VGIT:



$\{G \curvearrowright L\}$ form a inverse system.

$\rightsquigarrow \{X/\!/_L G\} \rightsquigarrow \lim X/\!/_L G$ is canonical.

call the limit quotient.

Another strategy: by Kapranov

Fact: X proj var ($\subseteq \mathbb{P}^N$)

cycles on X deg d dim m one para. by a var. $\mathrm{Chow}_{d,m}(X)$ (the Chow var).

G: gp (not necessarily reductive)

$G \curvearrowright X$

$\Rightarrow \exists$ open $U \subseteq X$ s.t. $\forall x \in U$

$\overline{G\cdot x}$ has a fixed dim deg.

$\Rightarrow$ makes sense: $U/G \ni \mathrm{pt} \leftrightarrow \overline{G\cdot x},\ x \in U$

crazy step

$\Rightarrow$ take $\overline{U/G} \subseteq \mathrm{Chow}_{d,m}(X)$

$\overline{U/G} = X/\!/_{ch} G$ is called the Chow quotient.

Rmk: $X/\!/_{ch} G$ does not dep on the U we choose.

($\Leftarrow$ by flatness stratification)

Rmk: $X/\!/_{ch} G$ is not a cat. quotient.

e.g. $G = \mathbb{C}^* \curvearrowright \mathbb{P}^2$

$\lambda\cdot[x:y:z] = [\lambda x : \lambda^{-1} y : z]$

moment map:

$\varphi : \mathbb{P}^2 \to \mathbb{R}$ $(= \mathrm{Lie}(\mathbb{C}^*)^\vee$, real part$)$

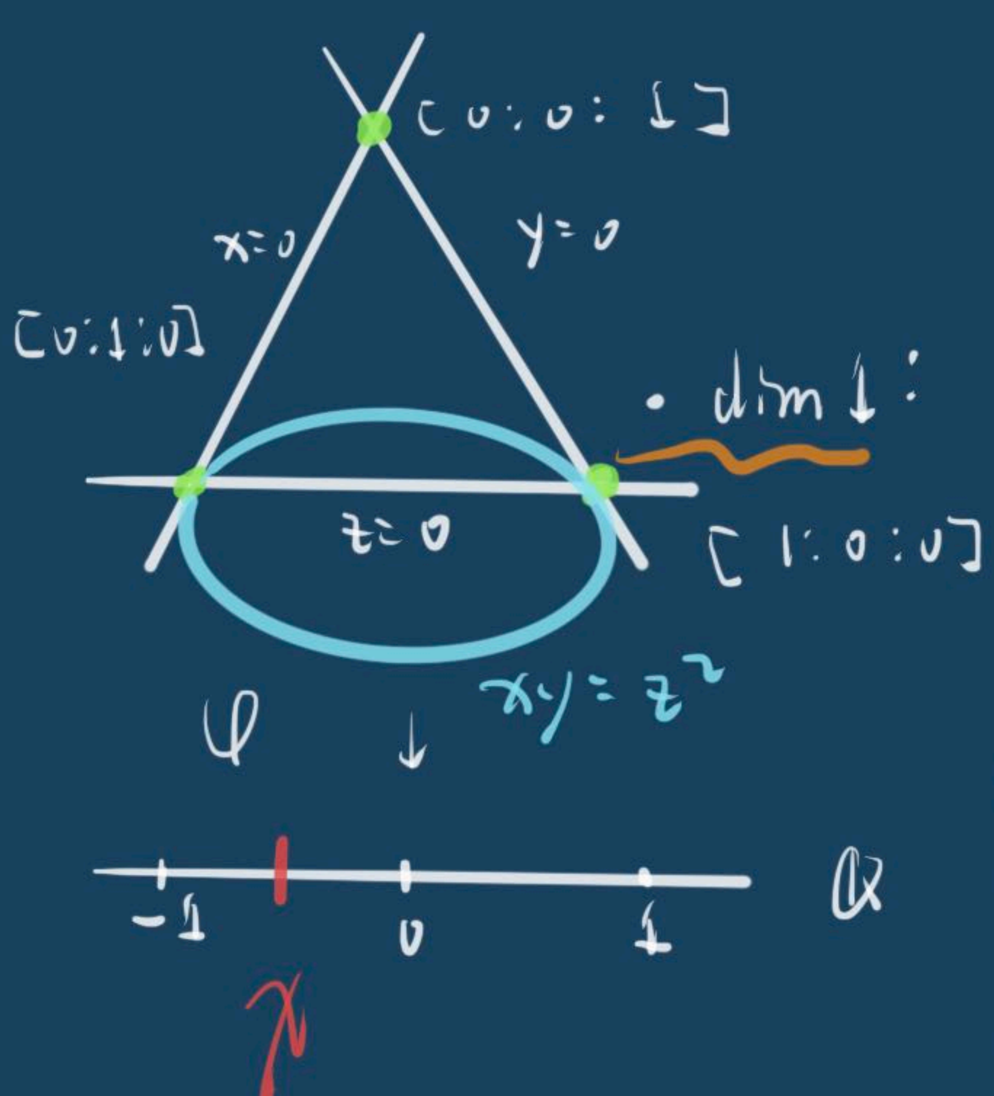
$$[x:y:z] \mapsto \frac{|x|^2-|y|^2}{|x|^2+|y|^2+|z|^2}$$

$\mathrm{im}\,\varphi = [-1, 1]$

G-orbits: · dim 1: $xy = az^2$, $a \in \mathbb{C}^*$

$[x:y:z] \neq [1:0:0],\ [0:1:0]$

$\mathrm{im}\,\varphi(xy = az^2) = (-1, 1)$



· dim 1: $x = 0$, $y = 0$, $z = 0$

minus coord. pts

l_x^* l_y^* l_z^*

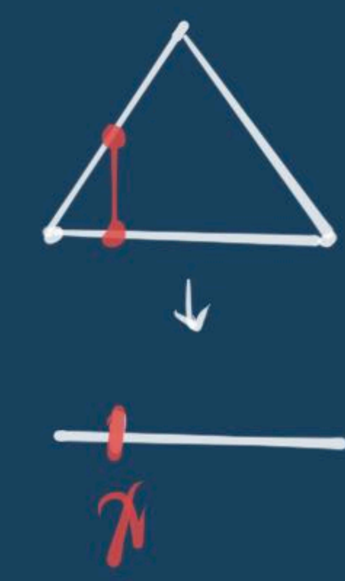
$\mathrm{im}\,\varphi(l_x^*) = (-1, 0)$

$\mathrm{im}\,\varphi(l_y^*) = (0, 1)$

$\mathrm{im}\,\varphi(l_z^*) = (-1, 1)$

· dim 0: $[1:0:0], [0:1:0], [0:0:1]$

$\mathrm{im}\,\varphi$: 1 -1 0



$\mathbb{P}^2 /\!/_\chi \mathbb{C}^* = \mathbb{P}^1$

p is s.s./s. wrt a chamber C

$\Updownarrow$

$\bar{C} \subseteq \varphi(\overline{G\cdot p})$ / $C \subseteq \varphi(G\cdot p)$

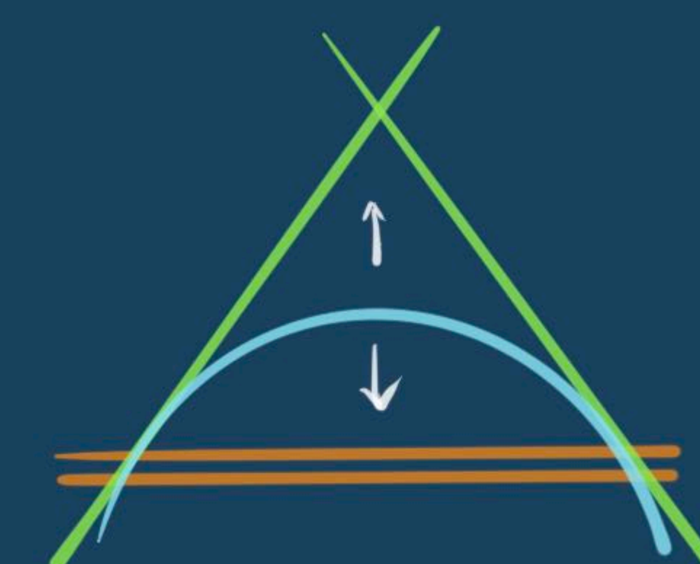
Chow quotient: look at generic

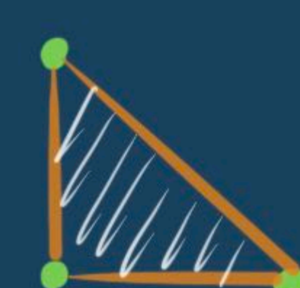
$xy = az^2$

$\Big\downarrow$ degenerations.

1) $a = 0$, $xy = 0$

2) $a = \infty$, $z^2 = 0$





pts in $\mathbb{P}^2 /\!/_{ch} \mathbb{C}^*$

$\Updownarrow$ 1:1

$\mathbb{C}^*$-inv. conics.

Chow quotient is $\mathbb{P}^1_a$.

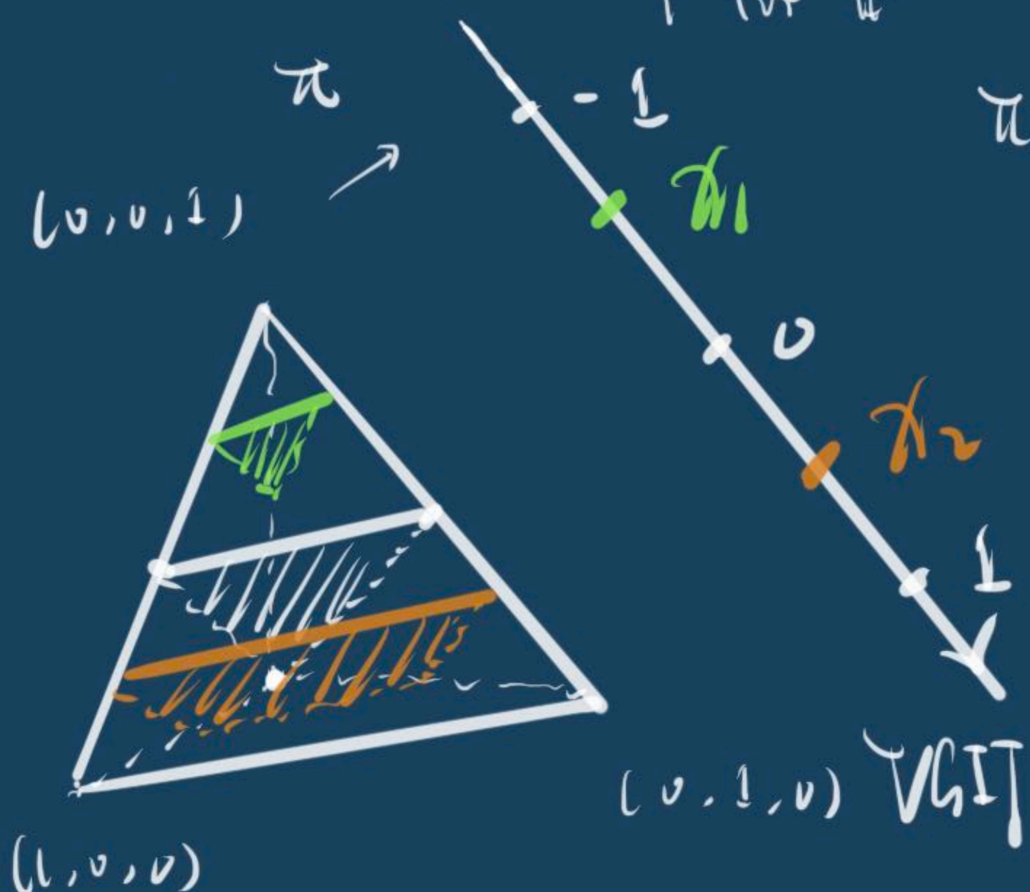
e.g. $H = \mathbb{C}^* \curvearrowright \mathbb{P}^3$, $L = \mathcal{O}(1)$

$\lambda[x:y:z:w] = [\lambda x : \lambda y : \lambda^{-1} z : w]$

$\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^3 \qquad \lambda \overset{i}{\mapsto} (\lambda, \lambda, \lambda^{-1})$

$\overset{\shortparallel}{T}$ for $\mathbb{P}^3$ $\qquad M_i : (1, 1, -1)$

$\pi = M_i^* : (x, y, z) \to (x, y, z)\begin{pmatrix}1\\1\\-1\end{pmatrix} = x + y - z$

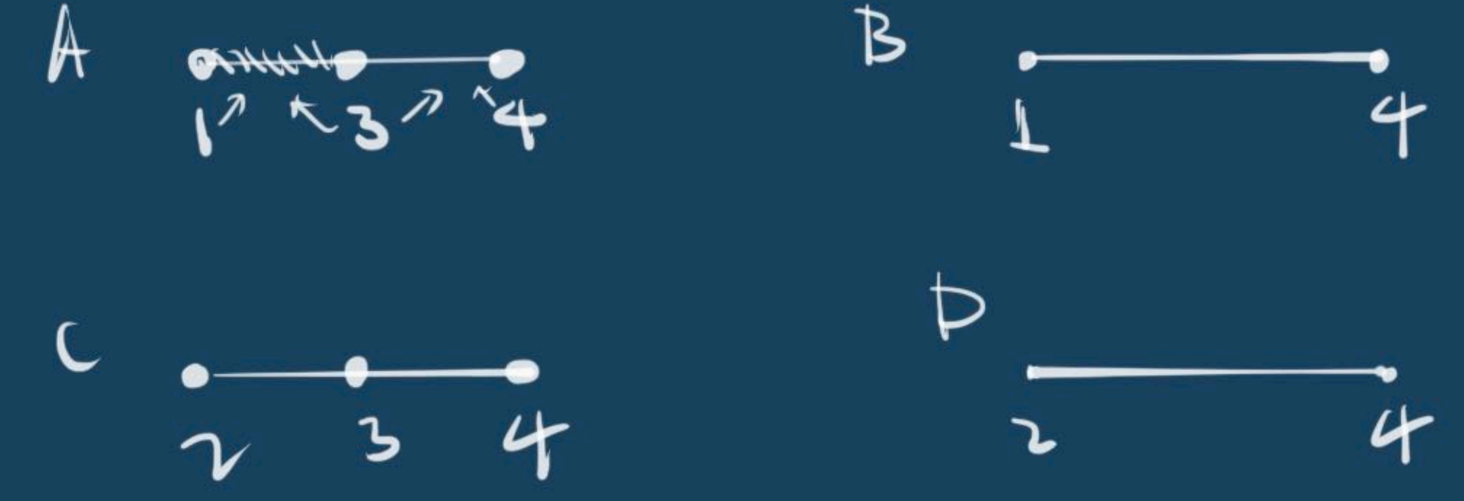


$$\begin{array}{c} M_{T,\mathbb{R}} \\ \downarrow M_i^* \\ M_{H,\mathbb{R}} \end{array}$$

VGIT: $\mathbb{P}^3 /\!/_{\chi_1} \mathbb{C}^* = \mathbb{P}^2 \qquad \mathbb{P}^3 /\!/_{\chi_2} \mathbb{C}^* = \mathbb{F}_1 (= \mathrm{Bl}_+ \mathbb{P}^2)$

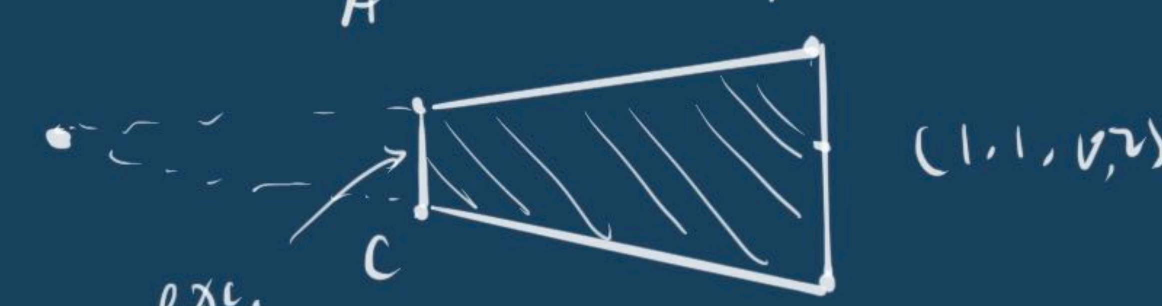
$\downarrow$

$\mathbb{P}^3 /\!/_{\chi} \mathbb{C}^* \quad \forall$ other $\chi \neq \chi$



For A:

pt:	1	2	3	4
coord of A:	(1	0	2	1)
C:	(0	1	2	1)
B:	(2	0	0	2)
D:	(0	2	0	2)



$\downarrow$

$\mathbb{F}_1$

given by this polytope

Answer for $\mathbb{P}^3 /\!/_{ch} \mathbb{C}^* = \mathbb{F}_1$.

Chow quotient: another strategy also gives $\mathbb{P}^3 /\!/_{ch} \mathbb{C}^*$ by using the secondary polytope.

look at all Δ'tions of

(actually not, but true here)

induced by π

Δ'tion $\leftrightarrow$ pt

coord. $= \sum$ vol of simplices containing it.

Given a lattice polytope Q (may be w/ a pt set $\mathcal{A}$ s.t. $Q = \mathrm{Conv}(\mathcal{A})$).

$\rightsquigarrow$ new polytope

vertices $\longleftrightarrow$ "some" Δ'tions of Q

$\uparrow$ regular

coord. given by $\sum_{\Delta \text{ simplex}} \mathrm{Vol}(\Delta)$

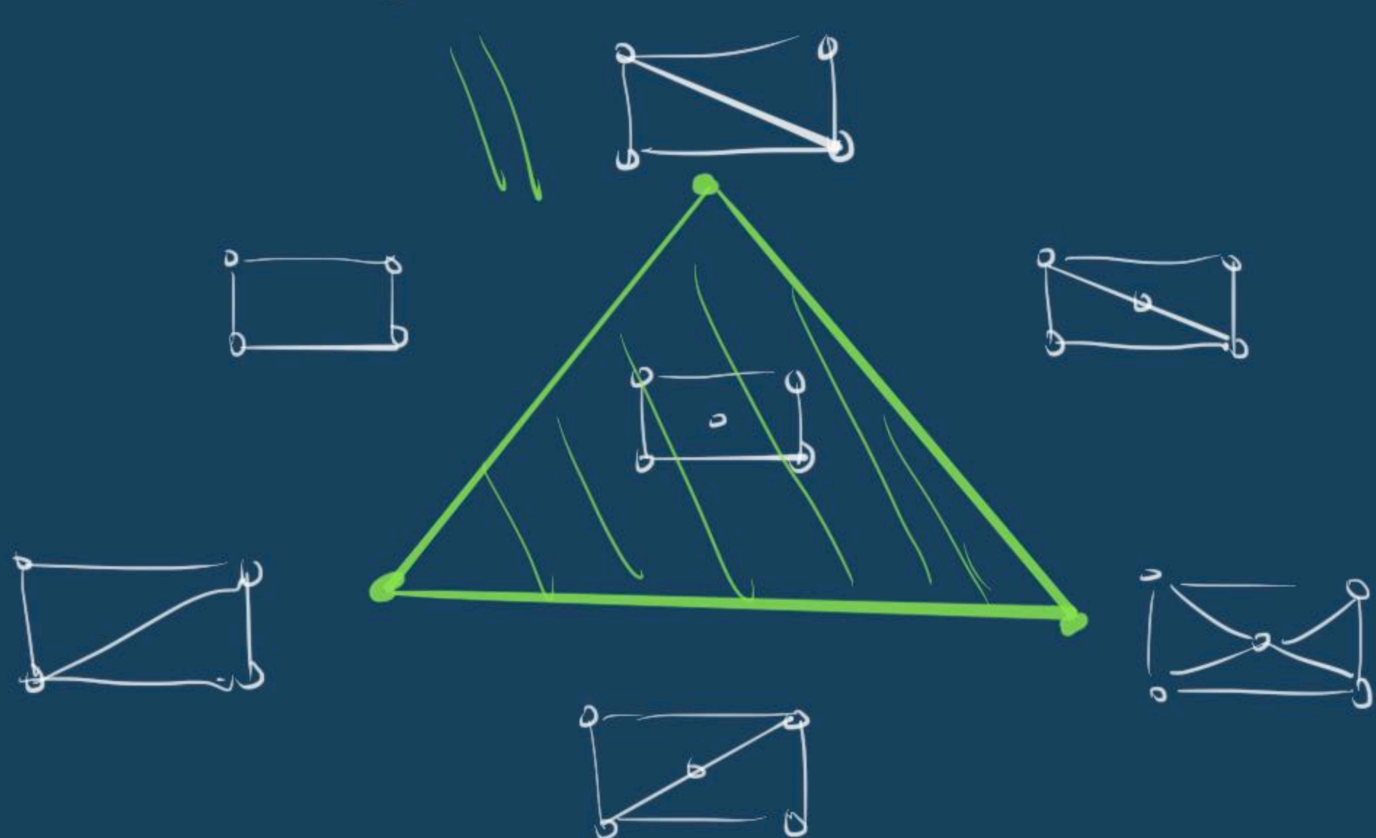
by using pts in $\mathcal{A}$.

introduced by Gelfand-Kapranov-Zelevinsky

This new polytope is called the secondary polytope, denoted by $\mathrm{Sec}(Q, \mathcal{A})$

e.g. $\mathcal{A} =$  $Q = \mathrm{Conv}(\mathcal{A})$

$\mathrm{Sec}(Q, \mathcal{A})$.



$X = \mathbb{P}^n \supseteq T$ big torus

$\Updownarrow \qquad \cup\!\!|$

$\Delta_n \qquad H$

$H \overset{i}{\hookrightarrow} T$

$H \curvearrowright X$

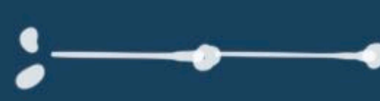
$$M_{T,\mathbb{R}} \supseteq \Delta_n \quad \big\downarrow i^*_{\mathbb{R}} \quad M_{H,\mathbb{R}}$$

Thm. $\mathbb{P}^n /\!/_{ch} H$ is also a polarized toric var.

$\Updownarrow$

Q polytope.

$Q = \mathrm{Sec}(i^*_{\mathbb{R}} \Delta_n)$

Rmk: Q should be viewed as a polytope w/ mult. faces. e.g. 

e.g. $\underset{\overset{\|}{T}}{(\mathbb{C}^*)^3 /_{\mathrm{diag}\,\mathbb{C}^*}} \curvearrowright \mathbb{P}^5$

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \curvearrowright [x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{34}]$

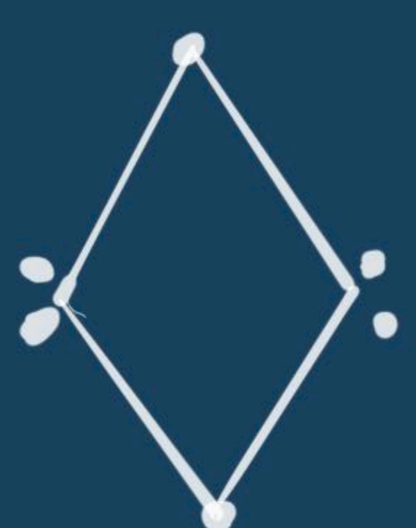
by $\lambda_i \cdot \lambda_j \cdot x_{ij}$

let $\lambda_3 = \lambda_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

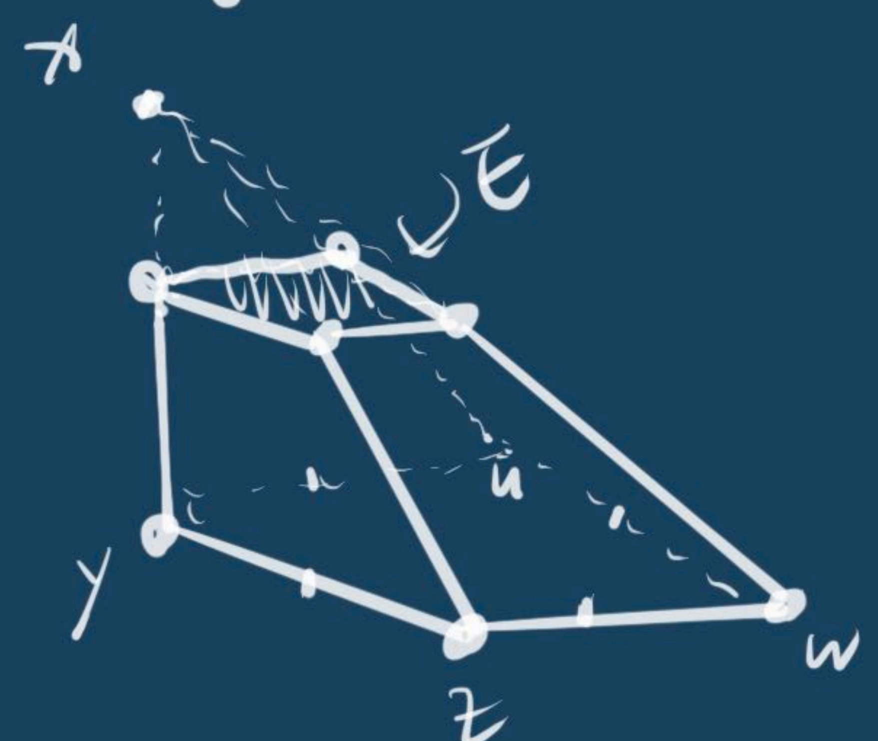
im $\Delta_5 = \mathrm{Conv}\{(1,1,0), (1,0,1), (0,1,1), (0,0,2)\}$

but $(1,1,0)$, $(0,0,2)$ have mult. 2.



$\rightsquigarrow$ eight Δ'tions (all of them regular)

$\rightsquigarrow$ polytope



$\mathbb{P}^5 /\!/_{ch} T \leftrightarrow Q$

alg: $Bl_1 \mathrm{Proj} \dfrac{\mathbb{C}[x, y, z, w, u]}{yw - uz}$

polarization: $\pi^*(\mathcal{O}(2)) \otimes \mathcal{O}(-E)$

Rmk: $\mathrm{Sec}(Q)$: $\Delta_n \xrightarrow{\pi} Q$

generalization: $P \xrightarrow{\pi} Q$

Billera-Sturmfels: $\Sigma(P \xrightarrow{\pi} Q)$ is call the fiber polytope

$\| \quad \dfrac{1}{\mathrm{Vol}(Q)} \displaystyle\int_{x \in Q} \pi^{-1}(x)\, dx.$

Thm: X pol. toric var $\leftrightarrow P$.

$H \overset{i}{\hookrightarrow} T \subseteq X$

$i^* P = Q$.

$X /\!/_{ch} H \leftrightarrow \Sigma(P \xrightarrow{\pi} Q)$

Moduli of hyperplane arrangement.

Set up: n hyperplanes in $\mathbb{P}^{k-1}$ in a general position

$$\begin{array}{ccc} k\left(\begin{matrix} * & & * \\ \vdots & & \vdots \\ * & & * \end{matrix}\right) \\ \uparrow & n & \uparrow \\ \mathbb{C}^* & & \mathbb{C}^* \end{array}$$

general position $\Leftrightarrow$ no $k\times k$ det $= 0$.

(= all Plücker coord $\neq 0$)

the set of such matrices $=: M^\circ_{k\times n}$

$(\mathbb{C}^*)^n \curvearrowright M^\circ_{k\times n}$ rescale each col.

$\rightsquigarrow$ n hyperplanes in $\mathbb{P}^{k-1}$.
(pts)

Note: $\exists$ a diag. $\mathbb{C}^* \curvearrowright$ trivially

$T = (\mathbb{C}^*)^n / \mathrm{diag}(\mathbb{C}^*) \rightsquigarrow M^\circ_{k\times n}/T = (\mathbb{P}^{k-1})^n_{gen} \cdots (A)$

$GL_k \curvearrowright M^\circ_{k\times n} \rightsquigarrow GL_k \backslash M^\circ_{k\times n} = Gr^\circ(k,n) \cdots (B)$

(A) : need kill aut's $\mathbb{P}^{k-1}$.

(B) : 〃 kill rescaling.

Thm (Gel'fand – Macpherson)

$$\begin{tikzcd} & M^\circ_{k\times n} \arrow[dl, "GL_k \backslash"'] \arrow[dr, "/T"] & \\ Gr^\circ(k,n) \arrow[d, "/T"'] & & (\mathbb{P}^{k-1})^n_{gen} \arrow[d, "/PGL_k"] \\ T\backslash Gr^\circ(k,n) & \cong & (\mathbb{P}^{k-1})^n_{gen}/PGL_k \end{tikzcd}$$

$\uparrow$ translated question | $\uparrow$ original questions

toric var's are involved

pt $\in T\backslash Gr^\circ(k,n)$

$\updownarrow$

T-orb. closure, i.e. a polarized toric var.

Q: Which one? (which polytope)

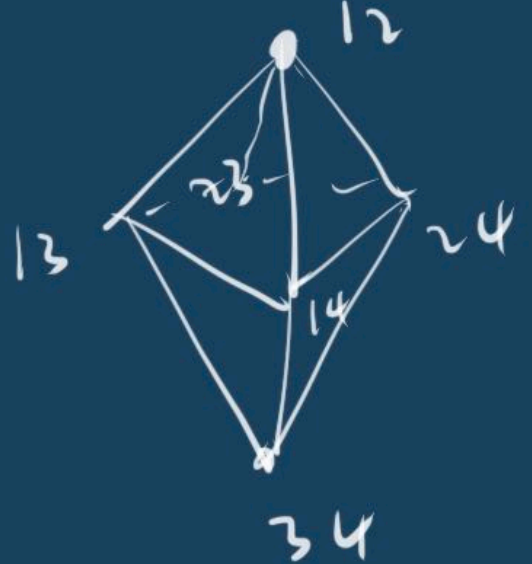
A: computed by moment map.

the answer is the hypersimplex:

$$\Delta_{k,n} = \{ \underset{\substack{\shortparallel \\ (x_1,\dots,x_n)}}{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = k \ , \ 0 \le x_i \le 1 \}.$$

Rmk: $k=1$ $\quad \Delta_{k,n} = \Delta_n$ simplex

$\Delta_{2,4}$:



$12 = (1\ 1\ 0\ 0)$

$34 = (0\ 0\ 1\ 1)$

Note: $\dim \Delta_{k,n} = n-1$

faces of $\Delta_{k,n}$ are also hypersimplices.

$2n$ facets: n: $x_i = 0$ $\qquad$ n: $x_i = 1$

$\Delta_{k,n-1}$ $\qquad$ $\Delta_{k-1,n-1}$.

Q: $T \backslash Gr^{\circ}(k,n)$ is not complete.

Goal: given a compactification

by taking the Chow quot.

U/G, $U = Gr^{\circ}(k,n)$

$G = T$

So we have: $Gr(k,n) /\!/_{ch} T$

Prop. (Kapranov).

Gel'fand - MacPherson works for Chow / GIT quotients.

$$Gr(k,n) /\!/_{ch} T \cong (\mathbb{P}^{k-1})^n /\!/ PGL_k(\mathbb{C})$$

Q: What are the fibers over the boundary pts in the chow quotient?

Observation:

e.g. $\mathrm{Gr}(2,4)/\!/_{ch}T \overset{Pl/T}{\hookrightarrow} \mathbb{P}^5/\!/_{ch}T$

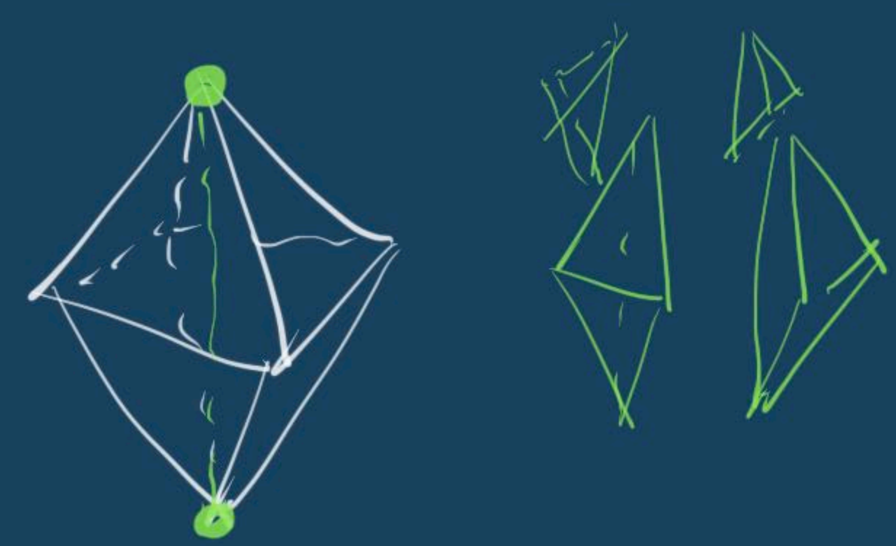
$T = (\mathbb{C}^*)^4 / \mathrm{diag}\, \mathbb{C}^*$ $\quad \begin{pmatrix} * & * & * & * \\ * & * & * & * \end{pmatrix} M_{2\times 4}$

4 pts on $\mathbb{P}^1$

$\dim(\mathbb{P}^5/\!/_{ch}T) = 5-3 = 2$

$\uparrow$ is toric

$\mathrm{Sec}(\Delta_{2,4})$



3 such pairs of opposite vertices

$\Updownarrow$

3 regular Δ'tions.

$\Downarrow$

3 vertices of $\mathrm{Sec}(\Delta_{2,4})$

$\Downarrow$

$\mathrm{Sec}(\Delta_{2,4}) =$ 

$\Updownarrow$

$\mathbb{P}^5/\!/_{ch}T = \mathbb{P}^2$ cut by Pl. relation deg 2.

$g = \frac{(d-1)(d-2)}{2} = 0 \Rightarrow \mathrm{Gr}(2,4)/\!/_{ch}T = \mathbb{P}^1.$