Toric var's.

Combinatorics vs. AG

Comb. $\xrightarrow{\text{help}}$ AG

- Toric Geometry (or more general: spherical) (started by Demazure)

  see almost all information from fans/poly

  e.g. degenerations of PPAV

  sometimes of CY's

- Tropical Geometry.

  alg var $\overset{\text{trop}}{\rightsquigarrow}$ trop. var

  e.g. dim 1 metric graphs

  e.g. $\mathbb{P}^2$, $3d-1$ pts set $\mathcal{A}$.

  # rat. curves through $\mathcal{A}$.

  $\shortparallel$

  formula by Kontsevich

  later ↑ is reformulated "easily" by trop (Gathmann-Markwig, Mikalkin).

- Boundedness of families

	antiample	$K_X = 0$	ample
w/ mild sing dim 2	: Alexeev.	×	Alexeev.
toric:	Borisov-Borisov ?		
dim≥3	: Birkar (BAB B)	$\boxed{?}$	Hacon-McKernan-Xu

Comb $\xleftarrow{\text{help}}$ AG

- McMullen's conj.

  Q: $\vec{h}=(h_1,\cdots,h_n)$ can this $\vec{h}$ be the f-vector of some simplicial poly?

  faces are simplex

  e.g.

  non.e.g.

  A: Stanley (Hard Lefschetz).

- Integral pts counting.

  kQ:

  $k\cdot$

  Q: how many $\mathbb{Z}$-pts in kQ?

  A: Ehrhart, Khovanskii-Pukhlikov, Brion

  $\top$

  Hirzebruch-Riemann-Roch

  # of $\mathbb{Z}$-pts $= \chi(L)$

- unimodular / log concave conj. of chromatic poly.

  Fix a graph G, color G by n diff colors, s.t. nbd different

  # of diff coloring = poly of n $= \chi(n)$ of deg $|\mathrm{Vert}(G)|$

  Whitney, Birkhoff.

  $\chi(n) = \sum a_i n^i$

  Thm (Adi-Huh-Katz)

  - $a_0 \le \cdots \le a_k \ge \cdots a_n$
  - $a_i^2 \ge a_{i-1} \cdot a_{i+1}$

  Ideal: graph $\rightsquigarrow$ matroid $\leftarrow$ on the matroid $\exists$ pol. Hodge structure.

Overview: (of toric var's) / $\bar{k}=k$ or just $\mathbb{C}$

Let $T = (\mathbb{C}^*)^n$, $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$

- as gp under " $\cdot$ "
- as a var $= V(x_1 \cdots x_n) \subset \mathbb{A}^n_{x_1,\dots,x_n}$

Two lattice $\cong \mathbb{Z}^n$

$N = \mathrm{Hom}(\mathbb{C}^*, T)$ $\quad$ $M = \mathrm{Hom}(T, \mathbb{C}^*)$

$N = M^\vee$ $\quad$ $M = N^\vee$

1-parameter subgp. $\quad$ character / monomials

$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ $\quad$ $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$

fans / cones glueing $\quad$ polyhedrons / polytopes

inverted picture $\leftarrow$ $\quad$ $\rightarrow$ direct picture

Benefits: from $\leftarrow$ or $\rightarrow$ we can see:

e.g. smooth / sing.

divisors / line bundles (ample?)

w/ no alg. compution too much.

cohomologies

quotients (GIT, Chow)

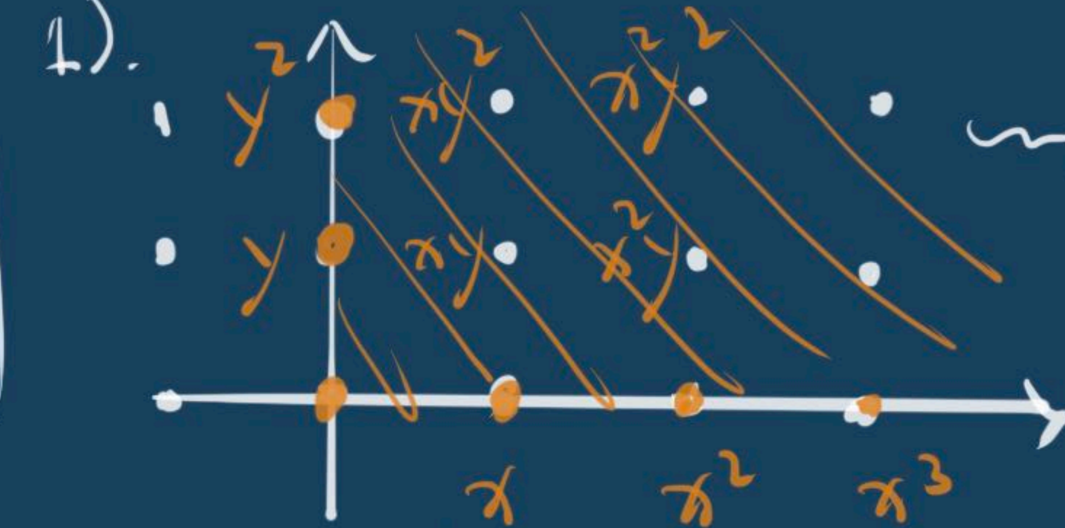
intersection #'s

Hodge #s

Warm up:

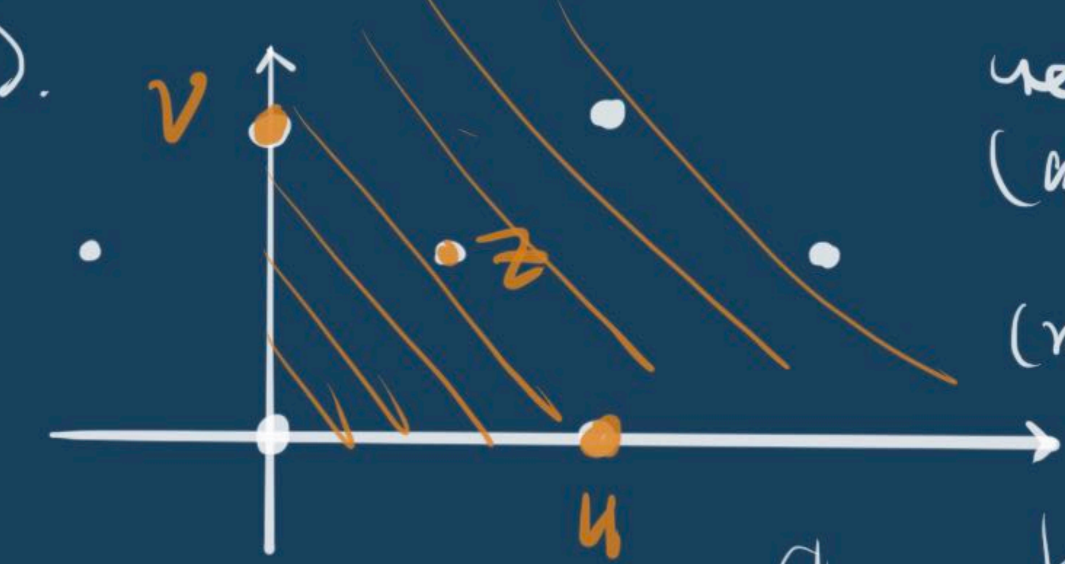
e.g. dim 2: $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$

1).



$\rightsquigarrow k[x,y] \rightsquigarrow \mathrm{Spec}\, k[x,y] = \mathbb{A}^2_{x,y}$

2).



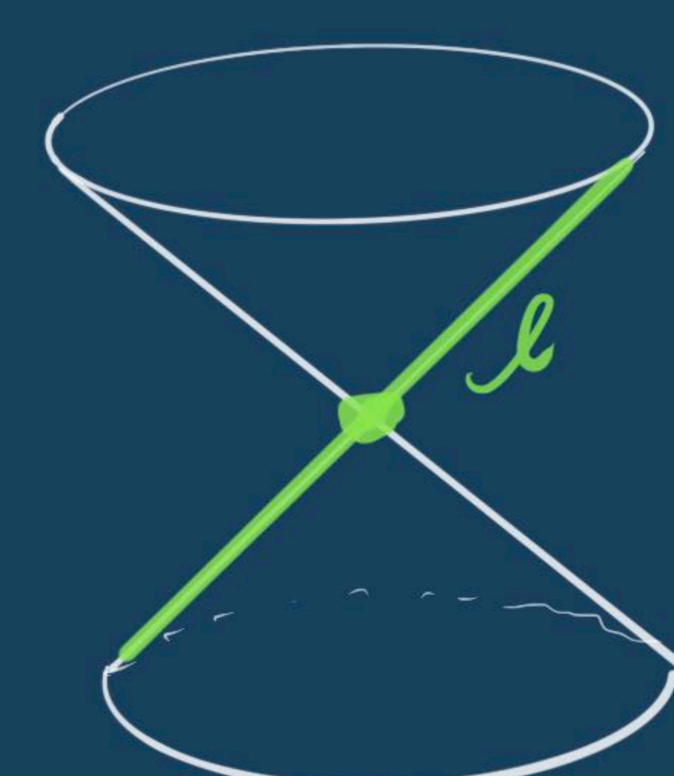
relation: $u+v=2z$ (add)

$\rightsquigarrow$ (mult.): $uv = z^2$

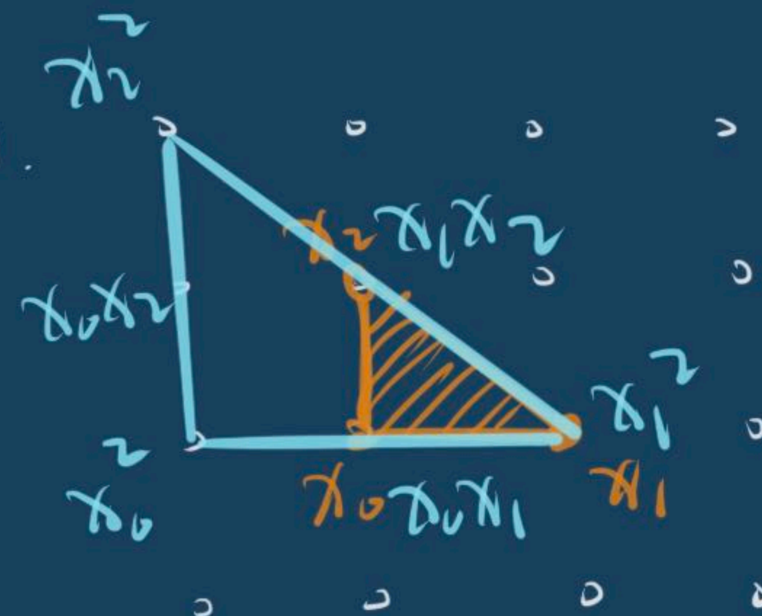
What is this? $\rightarrow$ $\mathrm{Spec}\, \dfrac{k[u,v,z]}{(uv-z^2)} = \mathrm{Spec}\, \dfrac{k[x,y,z]}{(x^2+y^2-z^2)}$

$x+y=u$

$x-y=v$

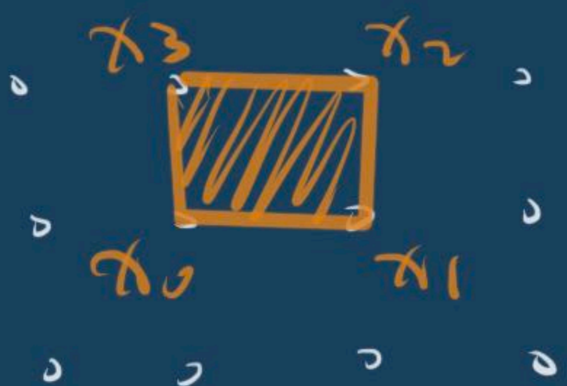


ℓ is a divisor which is Weil but not Cartier.

3). 

$\mathrm{Proj}\, k[x_0, x_1, x_2]$
$\parallel$
$(\mathbb{P}^2, \mathcal{O}(1))$

$\mathrm{Proj}\, k[x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2]$
$\parallel$
$(\mathbb{P}^2, \mathcal{O}(2))$

4). 

$$\mathrm{Proj}\, \frac{k[x_0, x_1, x_2, x_3]}{(x_0x_2 - x_1x_3)}$$
$\parallel$
$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$

rel (add)
$x_0 + x_2 = x_1 + x_3$
(mult):
$x_0x_2 = x_1x_3$

Affine Toric Var's:

$T = (k^*)^n$, as a var: $T = \mathbb{A}^n_k \setminus \bigcup_{i=1}^n \{x_i = 0\}$

e.g.  

As a group var: $\mu: T \times T \to T \quad (t_1, t_2) \mapsto t_1t_2$

$i: T \to T \quad t \mapsto t^{-1}$

$e: \mathrm{Spec}(k) \to T$

T satisfies

μ, i, e satisfy:

1). Associativity:

$$\begin{tikzcd} T \times T \times T \arrow[r, "{(\mu, id)}"] \arrow[d, "{(id, \mu)}"'] & T \times T \arrow[d, "\mu"] \\ T \times T \arrow[r, "\mu"] & T \end{tikzcd}$$
comm.

2). Identity:

$$\begin{tikzcd} \mathrm{Spec}(k) \times T \arrow[r, "{(e, id)}"] \arrow[dr, "\cong"'] & T \times T \arrow[d, "\mu"] & T \times \mathrm{Spec}(k) \arrow[l, "{(id, e)}"'] \arrow[dl, "\cong"] \\ & T & \end{tikzcd}$$
comm.

3). Inverse:

$$\begin{tikzcd} T \arrow[r, "{(i, id)}"] \arrow[d] & T \times T \arrow[d, "\mu"] & T \arrow[l, "{(id, i)}"'] \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "e"] & T & \mathrm{Spec}(k) \arrow[l, "e"'] \end{tikzcd}$$
comm.

Def: (affine toric vars)
X: var$/k$. aff. is (T-) toric if
$T \curvearrowright X$, $T \underset{\text{open dense}}{\subset} X$ s.t.
$T \curvearrowright T$ extends to $T \curvearrowright X$

Rmk: "open dense" $\Rightarrow$ irr.

 open not dense.

e.g. $T = (k^*)^3 = \operatorname{Spec} k[x_1^{\pm1}, x_2^{\pm1}, x_3^{\pm1}]$

1: $\mu: T\times T \to T$ $\qquad z_i = x_i y_i$

$(x_1,x_2,x_3),(y_1,y_2,y_3) \mapsto (z_1,z_2,z_3)$

$\mu^*: k[z_i^{\pm1}] \to k[x_i^{\pm1}] \otimes k[y_i^{\pm1}] \supseteq k[x_i^{\pm1}, y_i^{\pm1}]$

$z_i \mapsto x_i \otimes y_i \quad (= x_i y_i)$

2: $i: T \to T \qquad (x_1,x_2,x_3) \mapsto (x_1^{-1}, x_2^{-1}, x_3^{-1})$

$i^*: k[x_i^{\pm1}] \to k[x_i^{\pm1}]$

$x_i \mapsto x_i^{-1}$.

3. $e: \operatorname{Spec}(k) \to T$

$(0) = pt \mapsto (1,1,1)$.

$e^*: k[x_i^{\pm1}] \to k$

$x_i \mapsto 1$

Rmk: "$\otimes$" might be scary.

Remember: $A = k[a_i]/\mathfrak{a}$, $B = k[b_i]/\mathfrak{b}$

$A \otimes B = k[a_i, b_i]/(\mathfrak{a}, \mathfrak{b})$.

$T \subseteq X = \mathbb{A}^3_{x_1,x_2,x_3}$ $\quad$ extend $T \curvearrowright$

$T \curvearrowright X$ via $(\lambda_1,\lambda_2,\lambda_3)\cdot(x_1,x_2,x_3)$

$= (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$

i.e. $\lambda \cdot x = \lambda x$

$$\begin{tikzcd} \mathbb{A}^3 \arrow[r, "\lambda\cdot"] \arrow[dr, "\lambda^*\varphi"'] & \mathbb{A}^3 \arrow[d, "\varphi"] \\ & k \end{tikzcd}$$

$\varphi \in k[x_1,x_2,x_3]$

$\varphi = \sum a_m x^m \qquad \underset{m}{(m_1,m_2,m_3)}$

$\lambda^*\varphi = \sum a_m (\lambda\cdot x^m) = \sum a_m \lambda_1^{m_1}\lambda_2^{m_2}\lambda_3^{m_3}\, \underline{x_1^{m_1}x_2^{m_2}x_3^{m_3}}$

Orbits: $x_1x_2x_3 \neq 0$, T

$\sqcup\ x_1 = 0,\ x_2x_3 \neq 0$

$\sqcup\ x_2 = 0,\ x_1x_3 \neq 0$

$\sqcup\ x_3 = 0,\ x_1x_2 \neq 0$

$\sqcup\ x_1x_2 = 0,\ x_3 \neq 0$

$\sqcup\ x_1x_3 = 0,\ x_2 \neq 0$

$\sqcup\ x_2x_3 = 0,\ x_1 \neq 0$

$\sqcup\ x_1x_2x_3 = 0$

$\longleftrightarrow$

in $M_{\mathbb{R}} \subseteq \mathbb{R}^3$

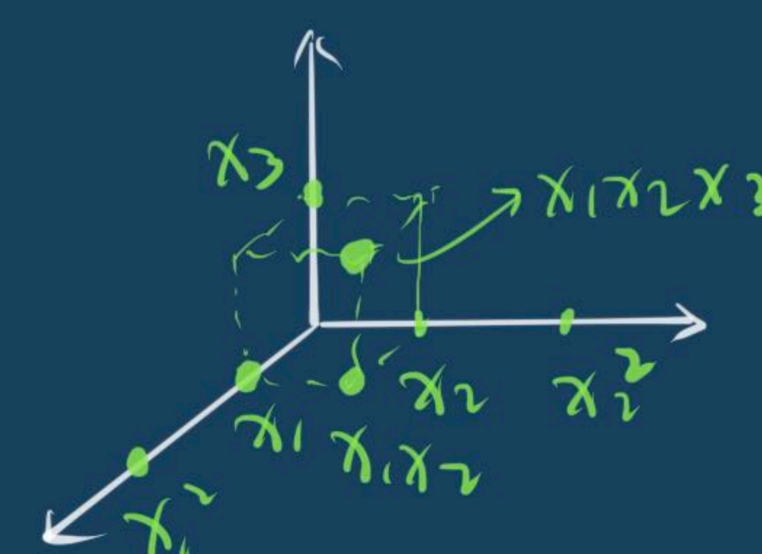
$G^\vee$ in $M_{\mathbb{R}}$

cone $= \{(a_1,a_2,a_3) \mid a_i \geq 0\}$

monomials of reg. functions on $\mathbb{A}^3_{x_1x_2x_3}$

$\updownarrow$ 1:1

$\mathbb{Z}$-pts in $G^\vee$

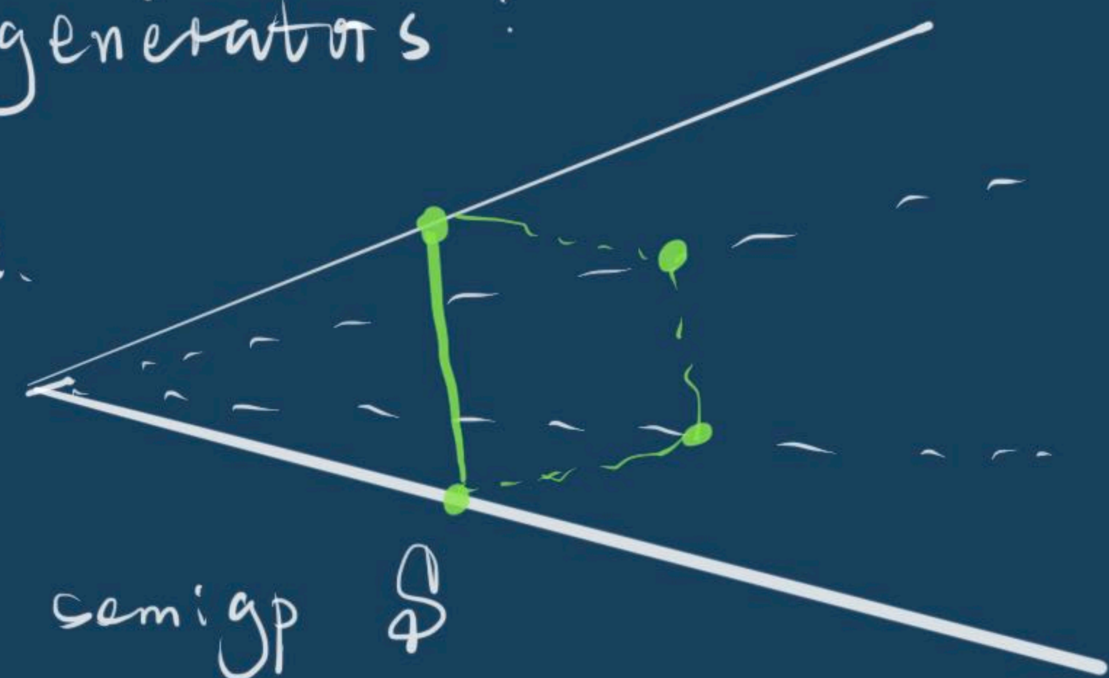


Spec $k[x_1,x_2,x_3]$

Note: all $\mathbb{Z}$-pts in $\sigma^\vee$ form a semigp
$$\overset{\|}{(S, +)}$$

More general: Take a finitely gen. cone $\sigma^\vee \subseteq M_{\mathbb{R}}$ (polyhedral)
w/ $\mathbb{Z}$-generators:

$$S = \sigma^\vee \cap M_{\mathbb{R}}.$$



Start from a semigp S

assume S: f.g. commutative

affine: $S \hookrightarrow \mathbb{Z}^m$.

Take $R = k[S] = \bigoplus_{m \in S} k\chi^m \quad (\chi^m \cdot \chi^{m'} = \chi^{m+m'})$

$X := \operatorname{Spec} R$

Thm: 1). X is an affine toric var.

2). All affine toric var's are of this form.

Pf: 1). $S \underset{\text{gen}}{\rightsquigarrow} \langle -S, S \rangle =: M$

thus $S \hookrightarrow M \rightsquigarrow \varphi: k[S] \hookrightarrow k[M]$

Then: $\operatorname{Spec}\varphi: \operatorname{Spec} k[M] \to \operatorname{Spec} k[S]$ is dominant.

(*) Recall: comm. alg: $A \underset{\varphi}{\hookrightarrow} B$
$$\Downarrow$$
($\ker\varphi \subseteq \overset{\text{nil}}{\sqrt{A}} \Leftrightarrow \operatorname{Spec} B \to \operatorname{Spec} A$ is dom.)

$$\operatorname{Spec} k[M] = \operatorname{Spec} k[S]\left[\frac{1}{\chi^{m_1}\cdots\chi^{m_k}}\right]$$
where $S = \langle m_1, \dots, m_k \rangle$
$$\| \qquad\qquad \|$$
$$T_M = D(\chi^{m_1}\cdots\chi^{m_k}) \subseteq \operatorname{Spec} k[S]$$

2). $T_M \curvearrowright X = \operatorname{Spec} A \qquad A$: f.g. k-alg no nilp.

$T_M \to \operatorname{Spec} A$ open dense thus dom.

by (*)

$k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \hookleftarrow A$
$$\Downarrow$$
A is M-graded

$\Downarrow$

A is gen. by a finite set in M

say $\{m_1, \dots, m_k\}$

$S := \langle m_1, \dots, m_k \rangle \Rightarrow$

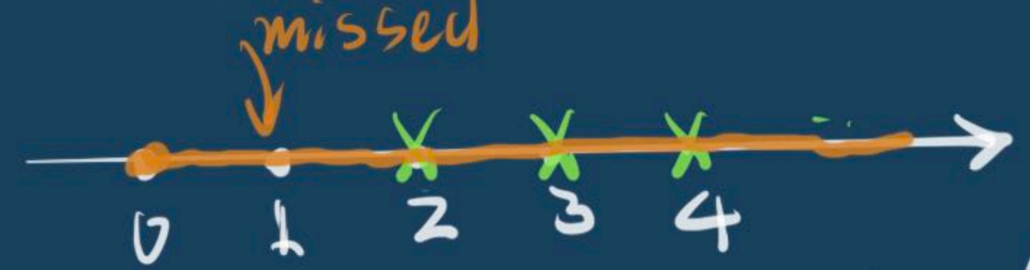
$X = \operatorname{Spec} k[S]$.

By the thm:

affine semigp $\overset{1:1}{\longleftrightarrow}$ affine toric var's.

But this is not "convenient"

e.g. $S=\{2,3,4,5\cdots\}\subseteq M\simeq\mathbb{Z}$

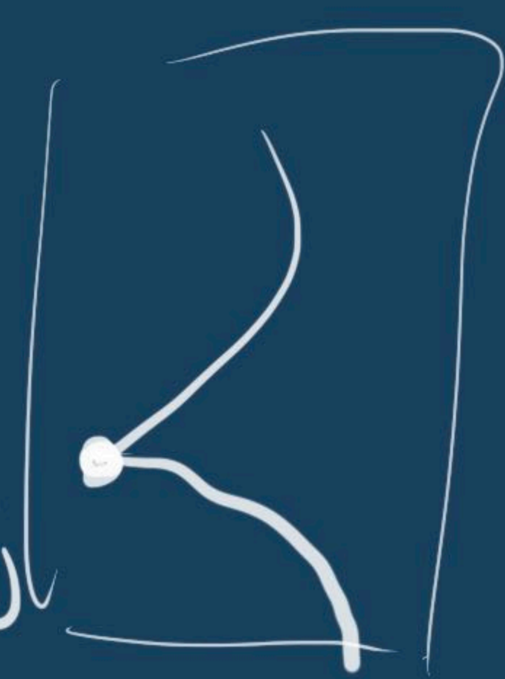


green pts are gen by $\{2,3\}$ (in S)

$\operatorname{Spec}k[t^2,t^3] = \operatorname{Spec}\dfrac{k[x,y]}{(y^3-x^2)}$

$x=t^2$

$y=t^3$



Don't want to make a choice, i.e. specify pts.

↓

Restrict our obj's i.e. not all affine toric var's

A: normal affine toric var's.

Dual cones:

Def: $N_\mathbb{R}\supset\sigma$ is a cone if

- $(0\in\sigma)$
- $x\in\sigma\Rightarrow\lambda x\in\sigma,\ \forall\lambda\in\mathbb{R}_{\geq 0}$

σ is convex is $x,y\in\sigma\Rightarrow ax+by\in\sigma$, $\forall a,b\in\mathbb{R}_+$.

$\Leftrightarrow$ (i.e. $\sigma+\sigma\subseteq\sigma$)

σ is strongly convex if

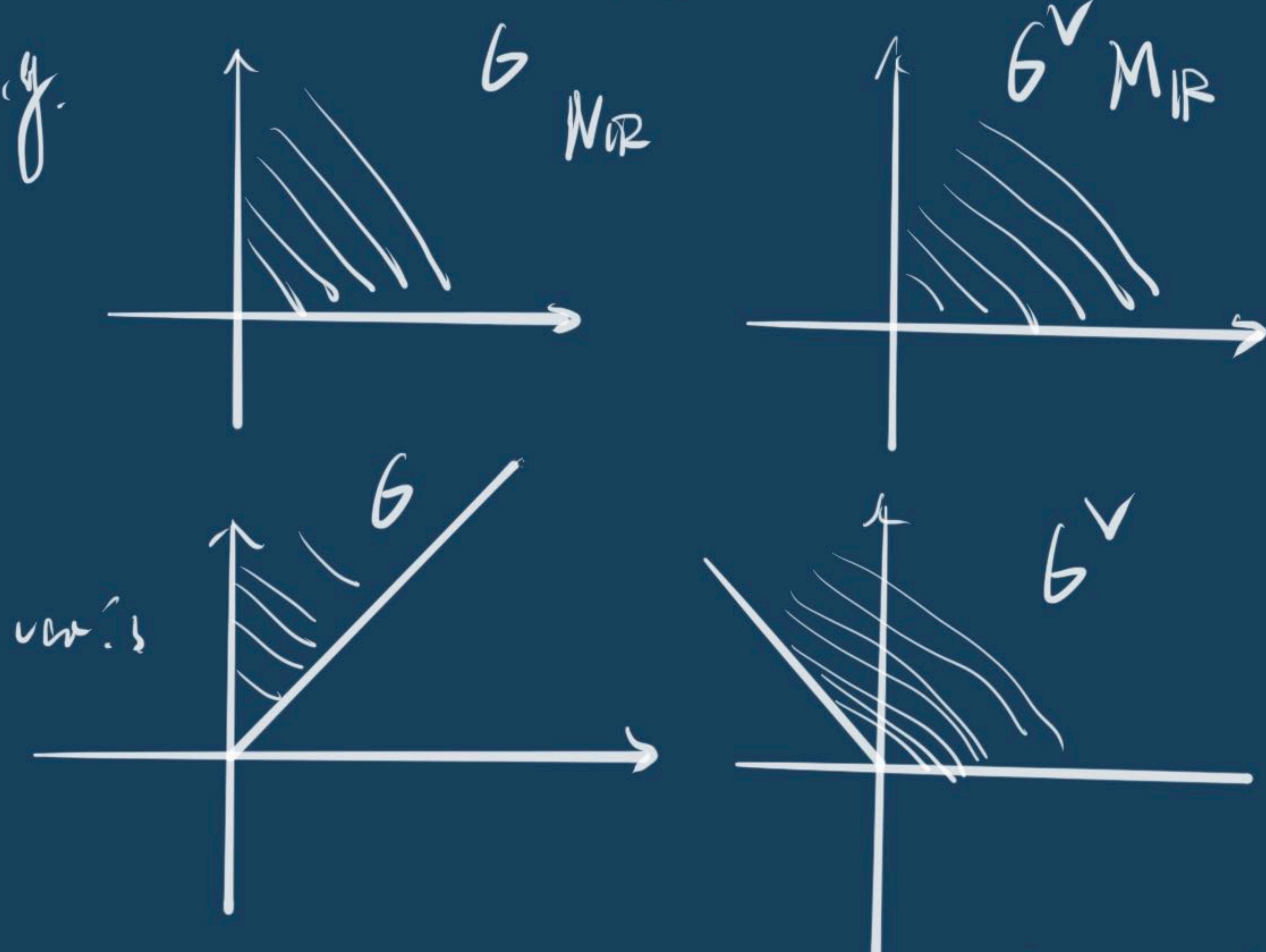
$\begin{cases}\sigma \text{ convex}\\ \sigma\cap(-\sigma)=\{0\}.\end{cases}$

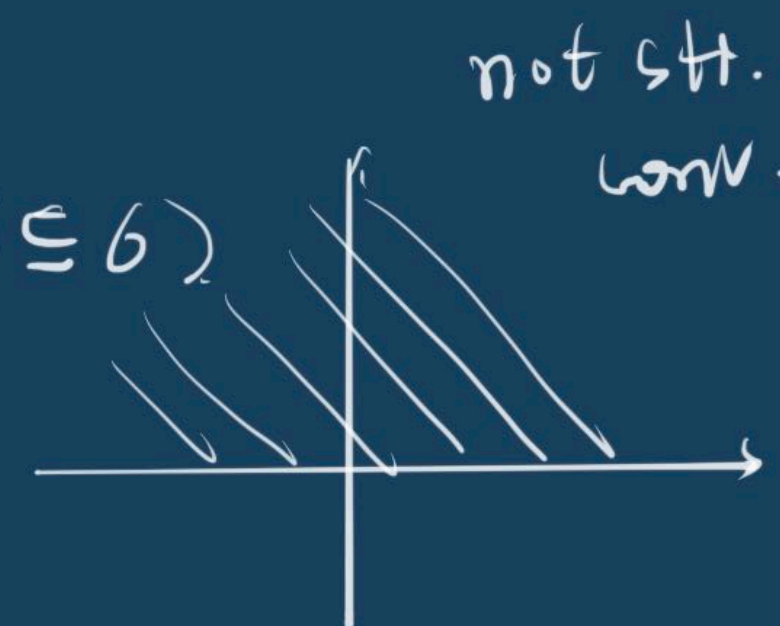
$\Updownarrow$

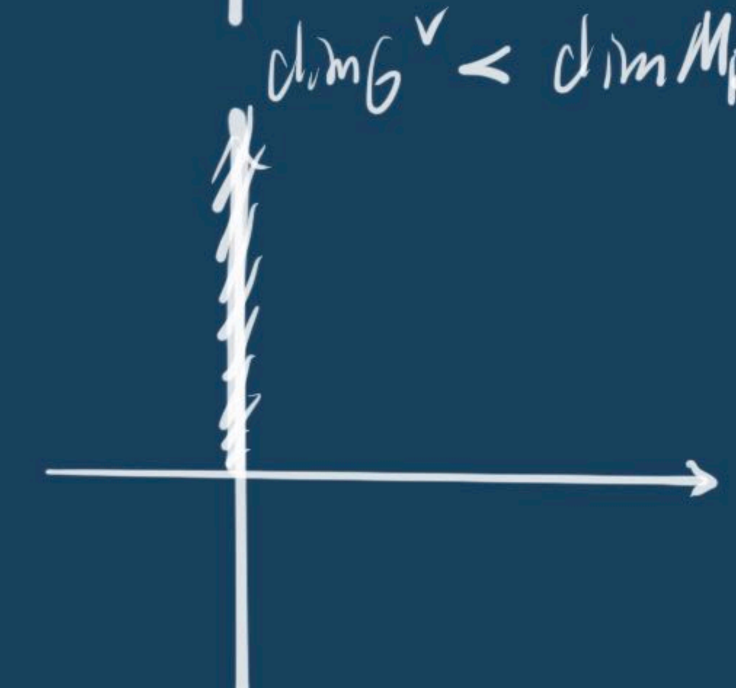
(i.e. no linear subspace $\overset{\neq\{0\}}{}\subseteq\sigma$)

Def: Given $\underset{N_\mathbb{R}}{\sigma}$ $\rightsquigarrow$ $\sigma^\vee=\{m\in M_\mathbb{R} \mid \langle m,y\rangle\geq 0,\ \forall y\in\sigma\}$, $\sigma^\vee\subseteq M_\mathbb{R}$

e.g.







Rmk: 1). $(\sigma^\vee)^\vee = \sigma$

2). σ strongly convex $\Leftrightarrow$ $\dim \sigma^\vee = \dim M_{\mathbb{R}}$.

$u \in \mathbb{R}^n$

$\downarrow$

$H_u^+ = \{ v \in \mathbb{R}^n \mid \langle u, v \rangle \geq 0 \}$

Thm: (Farkas).

$\mathbb{R}^n \neq \sigma \subseteq \mathbb{R}^n$ cone. f.g. poly.

facets of σ : (// codim 1 faces)

$\tau_i = H_{u_i} \cap \sigma, \quad i = 1, \cdots, k.$

$\Downarrow$

σ is an intersection of half spaces.

explicitly: $\sigma = \bigcap_{i=1}^{k} H_{u_i}^+$

How to interpret the thm?

$$\begin{array}{ccc} M_{\mathbb{R}} & \cong & \mathbb{R}^n \\ \cup & & \\ \sigma^\vee & = & \bigcap_{i=1}^{k} H_{u_i}^+ \end{array} \rightsquigarrow \text{coord. changing.}$$

$H_{u_i}^+ = \{ x_1 \geq 0 \} \supseteq \mathbb{Z}\text{-pts} \ (\cap M)$

monomials gen.

$\underline{x_1, x_2^{\pm 1}, x_3^{\pm 1} \cdots, x_n^{\pm 1}}$

$\Downarrow$

$A_i \cong k[x_1, \cdots x_n]_{x_2 \cdots x_n}$

expect: $\sigma^\vee$ might miss some $\mathbb{Z}$-pts

$\Downarrow$ not always.

S

$\searrow$

$k[S] = R = \bigcap A_i = \bigcap_{i=1}^{k} R_{S_i}$

Q: How to fix this?

look at only

A: normal vars.